Crossed modules and quantum groups in braided categories

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Abstract

Let A be a Hopf algebra in a braided category $\mathcal{C}$. Crossed modules over A are introduced and studied as objects with both module and comodule structures satisfying a compatibility condition. The category $\mathcal{DY}(\mathcal{C})_A^A$ of crossed modules is braided and is a concrete realization of a known general construction of a double or center of a monoidal category. For a quantum braided group $(A, \overline{A}, \mathcal{R})$ the corresponding braided category of modules $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ is identified with a full subcategory in $\mathcal{DY}(\mathcal{C})_A^A$. The connection with cross products is discussed and a suitable cross product in the class of quantum braided groups is built. Majid–Radford theorem, which gives equivalent conditions for an ordinary Hopf algebra to be such a cross product, is generalized to the braided category. Majid's bosonization theorem is also generalized.

1 Introduction

Crossed modules over a finite group G arisen in topology [38]. Crossed modules over groups and over Lie algebras were studied, as an algebraic object, in several contexts. In particular, see [6] in connection with cohomology of groups. A generalization for an arbitrary Hopf algebra A was noted by Yetter [40], who called the structures "crossed bimodules". This construction was extensively studied by Radford and Towber [34] under the name "Yetter-Drinfel'd structures". See also [14]. A crossed module is a vector space with both module and comodule structures over A satisfying a compatibility condition. The category ${}_A^A\mathcal{DY}$ of crossed modules is a convenient reformulation of the category of modules over Drinfel'd's quantum double $\mathcal{D}(A)$ [8] and has the corresponding braiding due to Drinfel'd. This was explained in [17] where also a functor embedding ${}_A\mathcal{M} \hookrightarrow {}_A^A\mathcal{DY}$ was introduced in the case where A is quasitriangular (a strict quantum group). One can obtain ${}_A^A\mathcal{DY}$ also as a 'center' or an 'inner double' of the monoidal category ${}_A\mathcal{M}$ of left modules by a general construction [26]. Crossed modules appear in

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different contexts related to Hopf algebras and quantum groups. For example, a subalgebra of left or right invariant forms in a bicovariant differential algebra over a Hopf algebra has a crossed module structure.

In this paper we introduce and study the category $\mathcal{DY}(\mathcal{C})^A_A$ of crossed modules over a Hopf algebra A living in an arbitrary braided monoidal category $\mathcal{C}$ (*braided-Hopf algebra* or *braided group* [18]-[21], [15]-[16], [27]-[30]). This category is also braided and most of the results for ordinary Hopf algebras hold in this situation. We also define the category $\mathcal{DY}(\mathcal{C})_{A,H}$ depending on bialgebra pairing $\rho:\ A\otimes H \to \underline{1}$, which is a fully braided analog of the category of modules $\mathcal{M}_{\mathcal{D}(A,H,\rho)}$ over Drinfel'd double $\mathcal{D}(A,H,\rho)$, and discuss (in a special case) the question when this category can be realized as a category of modules over something.

Braided groups have been extensively studied over the last few years and play an important role in q-deformed physics and mathematics [29]-[30]. Examples, applications and the basic theory of braided groups have been introduced and developed by Majid (some similar concepts arise independently in [15, 16] inspired by results on conformal field theory). In particular, Majid [27]-[28] defined a quantum braided group as a pair of a braided Hopf algebra H with a quasitriangular structure $\mathcal{R}$ satisfying axioms which are a generalization of ones for an ordinary quantum group [8], and a non-trivial class $\mathcal{O}$ of modules over H. He showed that the largest such class $\mathcal{C}_{\mathcal{O}(H)}$ is closed under tensor product and braided [28]; we will see that it becomes a full subcategory of $\mathcal{DY}(\mathcal{C})^H_H$ in the same way as for usual quantum groups in [17]. See also [22]. This embedding of modules over a quantum braided group into crossed modules is a key to applications to certain cross products and the bosonization construction [27],[23] and allows us to generalizes them to the fully braided setting.

Let A be a Hopf algebra in a braided category $\mathcal{C}$ and B a Hopf algebra in the category $\mathcal{DY}(\mathcal{C})^A_A$ of crossed modules. Then similarly to unbraided case [30] the tensor product $A\otimes B$ can be equipped with a natural cross product algebra and coalgebra $A \ltimes B$. The Majid-Radford theorem [23] gives equivalent condition for an ordinary Hopf algebra to be such a cross product; we show that a braided variant of this theorem holds when idempotents in a category $\mathcal{C}$ are split. The last condition is not essential because any braided category can be extended to one with split idempotents.

Similarly, let $(A,\mathcal{R}_A)$ be a quantum braided group in $\mathcal{C}$ and $(B,\mathcal{R}_B)$ a quantum braided group in $\mathcal{C}_{\mathcal{O}(H)}$. Then the cross product Hopf algebra $A \ltimes B$ in $\mathcal{C}$ is also a quantum braided group with $\mathcal{R}_{A\ltimes B}$ built from $\mathcal{R}_A$ and $\mathcal{R}_B$. This construction generalizes the Majid's bosonization procedure [27] which was defined in the case when A is an ordinary quantum group. Equivalent conditions for a quantum braided group to be a such cross product are also obtained.

Then we describe fully braided variant of Majid's transmutation procedure [19, 28] and show that analog of Majid's result [27] about relation between trans-

mutation and bosonization is true in our setting.

Finally, we suppose that a category $\mathcal{C}$ is balanced and define a ribbon structure on a quantum braided group $(A, \overline{A}, \mathcal{R})$ in $\mathcal{C}$ in a such way that the category $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ is also ribbon. We show that a ribbon structure is well-behaved under cross products and transmutation.

Results of this paper where announced in the note [3]. See also [2] where we work in the framework of ordinary Hopf algebras. The theory of crossed modules which is developed here is used to study of Hopf bimodules in [5]. Constructions of [4] illustrate our general theory.

An outline of the paper is as follows. In the preliminary Section 2 necessary notations connected with braided categories and braided groups are recalled from [29],[30]. In Section 3 the categories of crossed modules over a bialgebra (braided group) in a braided category $\mathcal{C}$ are introduced and studied. Section 4 is devoted to certain cross products of braided Hopf algebras and a braided variant of Majid-Radford theorem. Majid's definition of quantum braided group from [28] are discussed in Section 5. Finally, in Section 6 the results of Section 4 are extended to include cross products by quantum braided groups (generalizing [27] for cross products by braided groups with trivial $\mathcal{R}_A$), allowing us to prove a generalized bosonization theorem. Connections with transmutation and ribbon structure are discussed. Diagrammatic proofs of theorems are moved in appendix.

2 Preliminaries

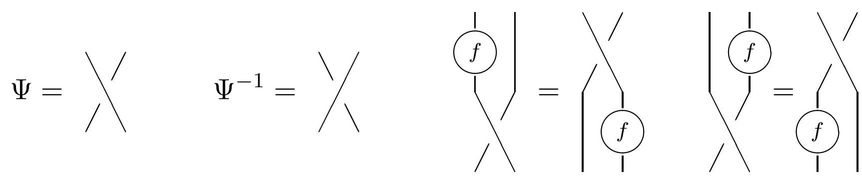
In this preliminary section we recall the basic notations and results of the theory of braided groups from [27]-[30]. See also [15],[16], and see [10],[12] in connection with braided categories themselves. We assume that the reader is familiar with ordinary Hopf algebras [37] and quantum groups [8].

2.1. Unless otherwise stated, we will suppose that $\mathcal{C} = (\mathcal{C}, \otimes, \underline{1}, \Psi)$ is *a braided (monoidal) category* with tensor product $\otimes$, unit object $\underline{1}$ and braiding Ψ. Without loss of generality by Mac Lane's coherence theorem [ML] we will assume that underlying monoidal category is strict, i.e. the functors $\_ \otimes (\_ \otimes \_)$ and $(\_ \otimes \_) \otimes \_$ coincide and $\underline{1} \otimes X = X = X \otimes \underline{1}$. A category $\mathcal{C}$ is called *pre-braided* if existence of Ψ^{-1} is not postulated.

A (braided) monoidal functor $F = (F, \lambda) : (\mathcal{C}, \otimes, \underline{1}, \Psi) \to (\mathcal{C}', \otimes', \underline{1}', \Psi')$ is a pair of a functor $F : \mathcal{C} \to \mathcal{C}'$ and isomorphism of functors $\lambda : F(\_) \otimes F(\_) \to F(\_ \otimes \_)$ which are compatible with braiding: $\lambda \circ \Psi' = F(\Psi) \circ \lambda$. We say that F is *strict* if λ is identity.

We actively use diagrammatic calculus in braided categories which is not a trivial generalization of 'wiring diagrams' for usual linear algebra (as in Penrose's

spin networks and [40]) developed by Majid ([27],[29]). Morphisms Ψ and Ψ^{-1} are represented by under and over crossing and algebraic information 'flows' along braids and tangles according to functoriality and the coherence theorem for braided categories [12]:



(2.1.1)

2.2. *An algebra* in a monoidal category $\mathcal{C}$ is an object A equipped with unit $\eta = \eta_A : \underline{1} \to A$ and multiplication $\mu = \mu_A : A \otimes A \to A$ obeying the axioms in Fig.1a. *A coalgebra* is object C equipped with counit $\epsilon = \epsilon_A : C \to \underline{1}$ and comultiplication $\Delta = \Delta_A : A \to A \otimes A$ obeying the axioms of algebra turned upside-down (Fig.1b). Let A, B be two algebras in a braided category $\mathcal{C}$. Then $A \otimes B$ is also an algebra with the multiplication

$$\mu_{A \otimes B} := (\mu_A \otimes \mu_B) \circ (A \otimes \Psi_{B,A} \otimes B) \tag{2.2.1}$$

Finally, [18],[28] *A bialgebra A in a braided category* $\mathcal{C}$ is an object in $\mathcal{C}$ equipped with algebra and coalgebra structures obeying the compatibility axioms in Fig.1c (i.e. Δ_A, ϵ_A are algebra homomorphisms or, equivalently, μ_A, η_A are coalgebra homomorphisms). *A Hopf algebra A in a braided category* $\mathcal{C}$ (*braided group* or *braided Hopf algebra*) is a bialgebra in $\mathcal{C}$ with antipode $S : A \to A$ which is convolution-inverse to identical map (the first identity in Fig.1d).

For example, in the category of sets with monoidal structure given by Cartesian product any object becomes a coalgebra uniquely, and a braided group in this category is an ordinary group. Mainly, we do not exploit additive structure of the category $\mathcal{C}$, and so we prefer Majid's term *braided group*. However, all results of this paper hold if we suppose categories and functors to be k-linear, where k is a commutative ring.

Now we recall basic properties of braided groups from [27]-[30]. Firstly, the last two identities in Fig.1d follow from the axioms [28]. Axioms for (right) module X over an algebra A with action $\mu_r : X \otimes A \to X$ in Fig.1e are obtained by "polarization" of the algebra axioms. We use slightly modified graphical notation for action to differ it from multiplication. The category $\mathcal{C}_A$ of right modules over bialgebra A in $\mathcal{C}$ is monoidal [28] with the module structure on the tensor product $X \otimes Y$ of modules X and Y defined using comultiplication (Fig.1e).

2.3. In diagrammatic language we have important symmetries [29] of the axioms of braided group which extend to symmetries of theorems when hypotheses are

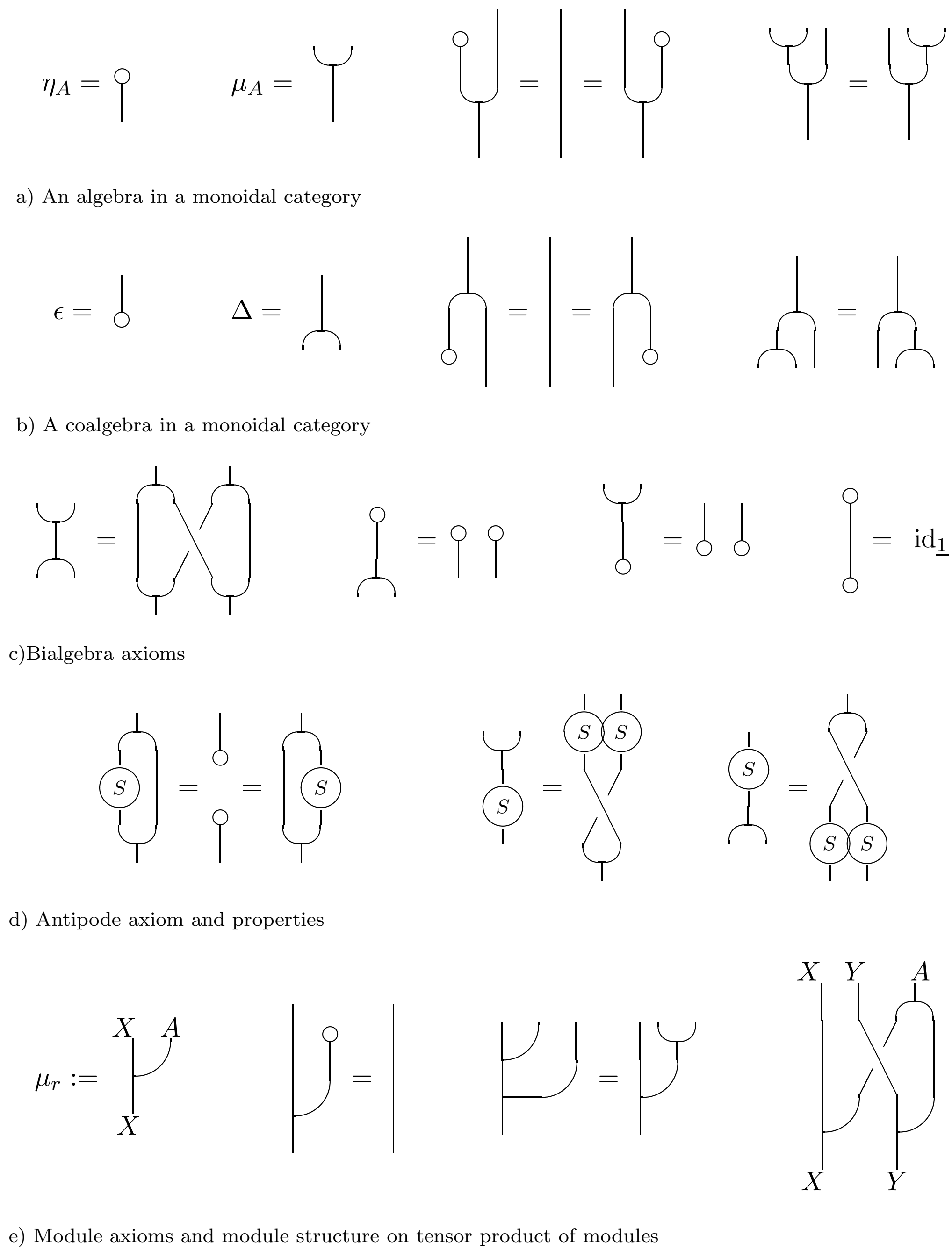


Figure 1: The basic algebraic structures in a braided category

also symmetric. For a diagram in a braided category $\mathcal{C}$ one can applies any of the following symmetry transformations: *input-output* or *up-side-down* which turns a structure into dual or co- structure, *left-right* and *mirror symmetry transformation*; the last is trivial in the unbraided case. For tangles the first two are axial symmetries and the third is reflection. Obtained diagram can be respectively attributed to the opposite category $\mathcal{C}^{\mathrm{op}}$ with the braiding $X \otimes Y \overset{\Psi_{Y,X}}{\longleftarrow} Y \otimes X$, or to the category $\mathcal{C}_{\mathrm{op}}$ with reversed tensor product and braiding

$$X \otimes_{\mathrm{op}} Y = Y \otimes X \overset{\Psi_{Y,X}}{\longrightarrow} X \otimes Y = Y \otimes_{\mathrm{op}} X,$$

or to the category $\overline{\mathcal{C}}$ with the same tensor product and with inverse braiding Ψ^{-1}. In particular, note that the collection of bialgebra (braided group) axioms for A in a braided category $\mathcal{C}$ is input-output and left-right symmetrical, i.e. A is a bialgebra (braided group) in the categories $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{C}_{\mathrm{op}}$ also.

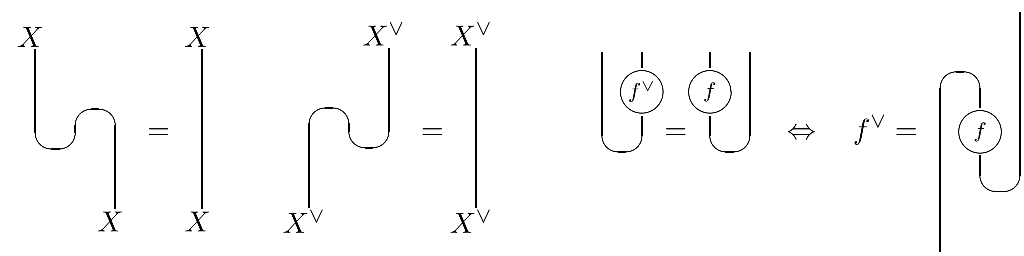
For any algebra (resp. coalgebra) A in $\mathcal{C}$ we will always consider *the opposite algebra* $(A^{\mathrm{op}}, \mu_{A^{\mathrm{op}}} := \mu_A \circ \Psi^{-1})$ (resp. *the opposite coalgebra* $(A_{\mathrm{op}}, \Delta_{A_{\mathrm{op}}} := \Psi^{-1} \circ \Delta_A)$ as an object of the category $\overline{\mathcal{C}}$. In particular, $(A^{\mathrm{op}})^{\mathrm{op}} = A$. If A is a bialgebra in $\mathcal{C}$ then A^{op} and A_{op} are bialgebras in $\overline{\mathcal{C}}$ (cf. [29]). Antipode S^- for A^{op} (or, the same, for A_{op}) is called *skew antipode* and equals S^{-1} if both S and S^- exist. The last two identities in Fig.1d mean exactly that antipode S_A is a bialgebra morphism $(A^{\mathrm{op}})_{\mathrm{op}} \to A$ (or $A \to (A_{\mathrm{op}})^{\mathrm{op}}$) in $\mathcal{C}$.

2.4. An object $X^\vee$ is called *(right) dual* for an object X in a (braided) monoidal category $\mathcal{C}$ if pairing $\cup :\ X \otimes X^\vee \to \underline{1}$ and copairing $\cap :\ \underline{1} \to X^\vee \otimes X$ obeying the first two identities in Fig.2a are given. Dual arrow $f^\vee$ is defined by one of the two equivalent conditions in Fig.2a. In this way a braided monoidal functor $(\_)^\vee :\ \mathcal{C} \to \mathcal{C}^{\mathrm{op}}_{\mathrm{op}}$ can be defined if $X^\vee$ exists for each $X \in \mathrm{Obj}(\mathcal{C})$. Without loss of generality by coherence theorem [13] we shall assume that $(\_)^\vee$ is a strict monoidal functor: $(X \otimes Y)^\vee = Y^\vee \otimes X^\vee$, $(f \otimes g)^\vee = g^\vee \otimes f^\vee$. Application of $(\_)^\vee$ to a bialgebra (braided group) A gives the same structure on $A^\vee$ (cf. [29]):

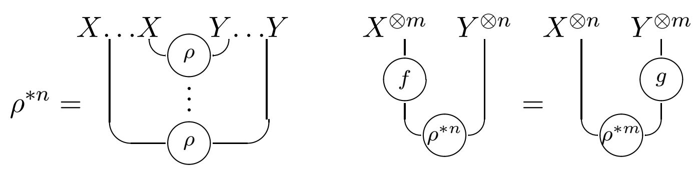
$$\mu_{A^\vee} := {\Delta_A}^\vee, \qquad \eta_{A^\vee} := {\epsilon_A}^\vee, \qquad \Delta_{A^\vee} := {\mu_A}^\vee, \qquad \epsilon_{A^\vee} := {\eta_A}^\vee \qquad (2.4.1)$$

Left dual ${}^\vee X$ is the right dual in the category $\mathcal{C}^{\mathrm{op}}$ or $\mathcal{C}_{\mathrm{op}}$.

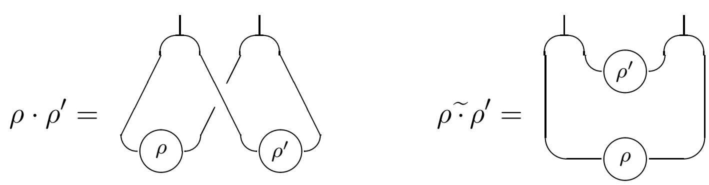
More generally, let $\mathcal{C}$ be a monoidal category, $X, Y \in \mathrm{Obj}(\mathcal{C})$, $\rho \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \underline{1})$ and morphisms $\rho^{*n} :\ X^{\otimes n} \otimes Y^{\otimes n} \to \underline{1}$ be defined by Fig.2b. We say that arrows $f :\ X^{\otimes m} \to X^{\otimes n}$ and $g :\ Y^{\otimes n} \to Y^{\otimes m}$ are *ρ-dual* if the identity in Fig.2b is satisfied. Let A and H be bialgebras in braided category $\mathcal{C}$. Morphism $\rho :\ A \otimes H \to \underline{1}$ is called *a bialgebra pairing* if algebra (resp. coalgebra) structure on A and coalgebra (resp. algebra) structure on H are ρ-dual. Convolution product $\cdot$ and 'the second' product $\tilde{\cdot}$ for $\rho, \rho' \in \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \underline{1})$ are defined in Fig.2c. We denote by ρ^-, $\rho^\sim$



a) (Right) dual object and arrow



b) ρ-dual arrows



c) Two products of pairings

Figure 2: Dual and pairings.

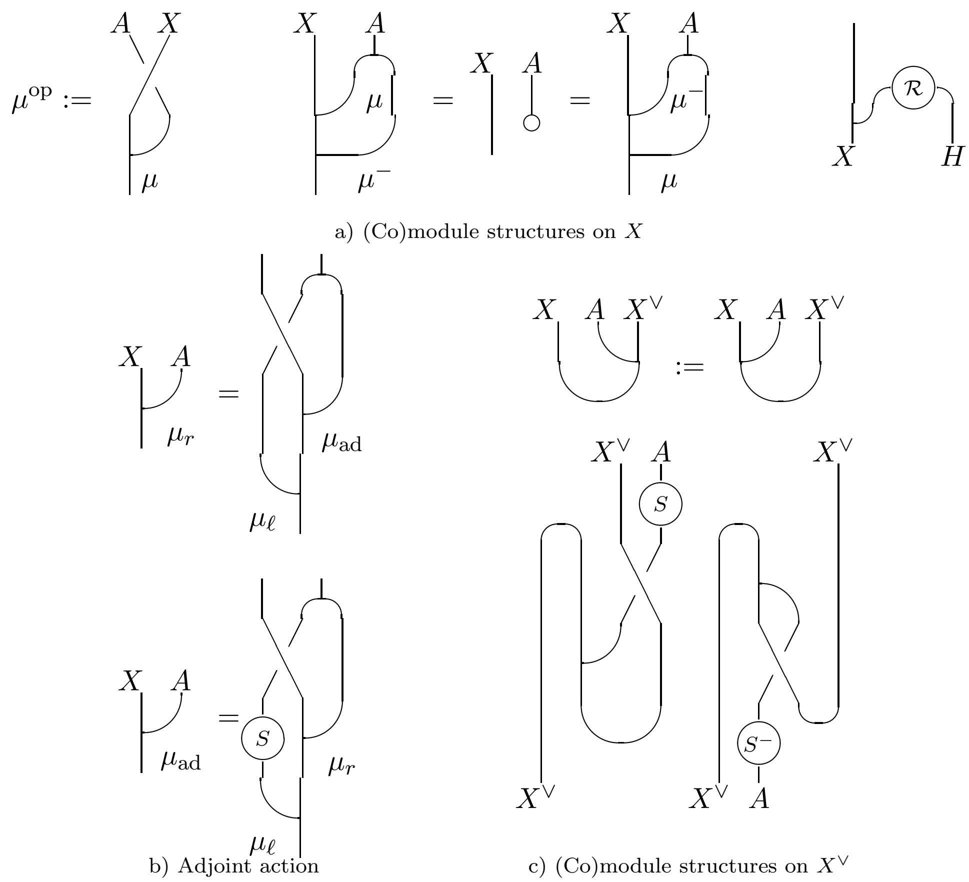


Figure 3: Constructions of new (co)modules

corresponding inverse to ρ. Let $\overline{\rho} := \rho^- \circ \Psi^{-1}$. If A or H has (skew) antipode then $\rho^\sim$ (resp. ρ^-) exists and

$$\rho \circ (S_A \otimes H) = \rho^\sim = \rho \circ (A \otimes S_H) \qquad \rho \circ (S_A^- \otimes H) = \rho^- = \rho \circ (A \otimes S_H^-) \quad (2.4.2)$$

If ρ^- or $\rho^\sim$ exists then ρ-duality between multiplications and comultiplications implies ρ-duality between units and counits. If (A, H, ρ) is bialgebra pairing in $\mathcal{C}$ then $(A_{\mathrm{op}}, H_{\mathrm{op}}, \rho^-)$, $(A^{\mathrm{op}}, H^{\mathrm{op}}, \rho^\sim)$, $(H^{\mathrm{op}}, A^{\mathrm{op}}, \overline{\rho})$ are bialgebra pairing in $\overline{\mathcal{C}}$.

A bialgebra copairing $\mathcal{R}$ in $\mathcal{C}$ is a bialgebra pairing in $\mathcal{C}^{\mathrm{op}}$. In this case we will say about $\mathcal{R}$-*codual morphisms* and *bialgebra copairing* and use similar notations $\mathcal{R}^-$, $\mathcal{R}^\sim$, $\overline{\mathcal{R}}$.

2.5. Let A be a bialgebra (braided group) in $\mathcal{C}$ and X a (co)module over A. Then there exist other various (co)module structures on X or $X^\vee$ over A or its opposite

or dual [29], which can be obtained as a result of sequential application of the following basic procedures (and their left-right, input-output and mirror reversed forms). One can define new (co)module structures on the underlying object of a right A-module X with action μ as shown in Fig.3a:

- left A^{op}-module with *opposite* action μ^{op} (this construction extends to isomorphism of monoidal categories $\mathcal{C}_A$ and ${}_{A^{\mathrm{op}}}\overline{\mathcal{C}}$);
- right A^{op}-module with *inverse* action μ^- if A is a braided group (and, therefore, μ^- exists and equals $\mu\circ(X\otimes S_A)$) (in this case one has a strict monoidal functor from $\mathcal{C}_A$ to $\mathcal{C}_{(A^{\mathrm{op}})_{\mathrm{op}}}$);
- right H-comodule via bialgebra copairing $\mathcal{R} : \underline{1} \to A\otimes H$ (this correspondence extends to strict monoidal functor $\mathcal{C}_A \to \overline{\mathcal{C}}^{H^{\mathrm{op}}}$).

To complete the picture we note that identity functor together with the natural transformation Ψ defines equivalence of the monoidal categories $\mathcal{C}_A$ and $(\overline{\mathcal{C}}_{A_{\mathrm{op}}})_{\mathrm{op}}$.

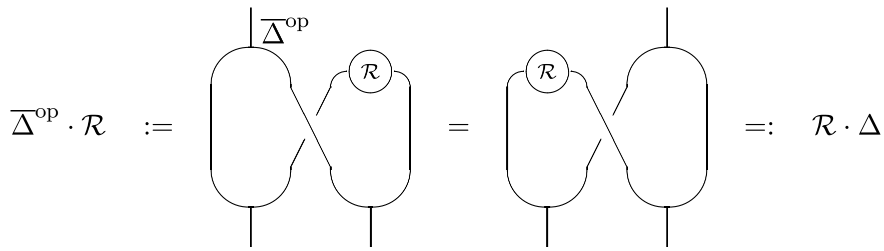
Let A be a bialgebra and (X, μ_ℓ, μ_r) a bimodule over A. An action μ_{ad} : $X\otimes A \to A$ is called (right) adjoint if the first identity in Fig.3b is satisfied. If A is a braided group adjoint action always exists and is defined by the second identity in Fig.3b. See [24] about properties of adjoint action. Adjoint coaction is defined by input-output reversed diagrams.

Let X be a right module over braided group A. The identity in Fig.3b defines left A-module structure on dual object $X^\vee$. Then one can use procedures described above to convert left action into right action. The last two diagrams in Fig.3c describe (co)module structure on $X^\vee$ which turns it into dual in the category $\mathcal{C}_H$ (resp. $\mathcal{C}^H$).

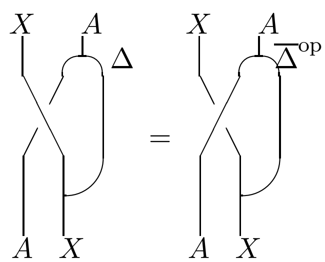
2.6. Quantum braided groups in a braided category were introduced in [28] and basic theory was developed there. The following are definitions from [28] in a slightly modified form suitable for our use.

A quasitriangular bialgebra in a braided category $\mathcal{C}$ is a pair of bialgebras A in $\mathcal{C}$ and $\overline{A}$ in $\overline{\mathcal{C}}$ with the same underlying algebra (Δ and $\overline{\Delta}$ are comultiplications in A and $\overline{A}$ respectively), and convolution invertible bialgebra copairing (*quasitriangular structure*) $\mathcal{R} : \underline{1} \to \overline{A}_{\mathrm{op}}\otimes A$, satisfying the condition in Fig.4a. (It follows directly from the definition that counits for A and for $\overline{A}$ are the same.) *A quantum braided group* or *a quasitriangular Hopf algebra* in $\mathcal{C}$ is a quasitriangular bialgebra such that A and $\overline{A}$ have antipodes S and $\overline{S}$ respectively. (In this case $\mathcal{R}^- = (\overline{S}\otimes A)\circ\mathcal{R}$ and $\mathcal{R}^\sim = (A\otimes S)\circ\mathcal{R}$.)

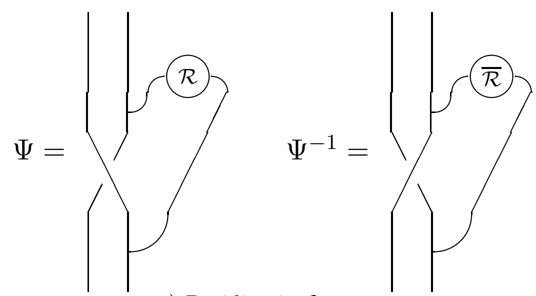
In particular, for any bialgebra (braided group) A the pair (A, A_{op}) is a quasitriangular bialgebra (quantum braided group) with the trivial quasitriangular structure $\mathcal{R} = \eta\otimes\eta$.



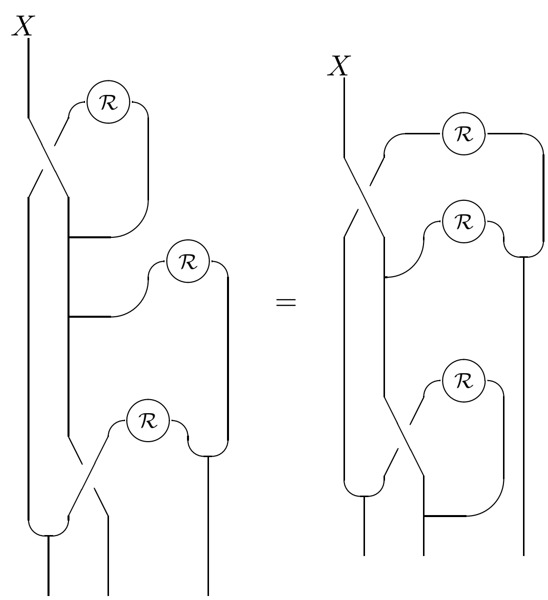
a) Comultiplications Δ and $\overline{\Delta}^{\rm op}$ are adjoint



b) The condition on modules from $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$



c) Braiding in $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$



d) A relative form of the Yang-Baxter equation.

Figure 4: The axiom for a quantum braided group $(A,\mathcal{R})$. The braided category $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ of modules.

Category $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ is a full subcategory of $\mathcal{C}_A$ with objects X satisfying the identity in Fig.4b. $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ is a monoidal subcategory of $\mathcal{C}_A$ and braided with Ψ and Ψ^{-1} shown in Fig.4c (see. [28] where corresponding categories of left modules are introduced and studied). We use a brief notation $\mathcal{C}_{\mathcal{O}(A)}$ for $\mathcal{C}_{\mathcal{O}(A,A_{\mathrm{op}})}$.

As Majid showed, the basic formulas for ordinary quantum groups [9] have analogues in this more general context but some of them exist only in a relative form: actions on arbitrary modules from $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ take part in them. One can obtain the standard formulas for ordinary quantum groups if one considers the action on the unit element of the regular module. As an example a relative form of the Yang-Baxter equation for $X \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}(A,\overline{A})})$ is shown on Fig.4d.

3 Crossed modules.

3.1. Here we introduce categories of crossed modules over a bialgebra (braided group) A in a braided category $\mathcal{C}$.

DEFINITION 3.1.1 A right *(resp.* left-right*)* crossed module *over a bialgebra A in a braided category $\mathcal{C}$ is an object X with right (resp. left) A-module and right A-comodule structures obeying the first (resp. the second) identity in Fig.5a. $\mathcal{DY}(\mathcal{C})_A^A$ (resp. ${}_A\mathcal{DY}(\mathcal{C})^A$) is the* category of right *(resp.* left-right*)* crossed modules *with morphisms which are both module and comodule maps. Objects of categories ${}_A^A\mathcal{DY}(\mathcal{C})$ and ${}^A\mathcal{DY}(\mathcal{C})_A$ are described by left-right reversed axioms.*

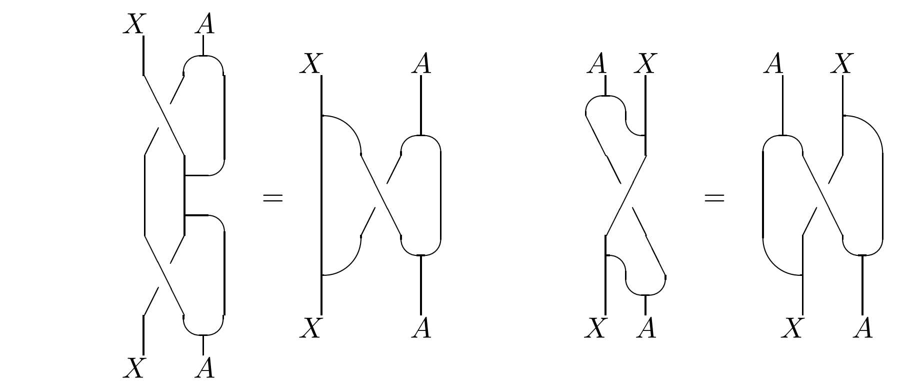
Note, that the category $\mathcal{DY}(\mathcal{C})_A^A$ is defined when $\mathcal{C}$ is only pre-braided, whereas definition of ${}_A\mathcal{DY}(\mathcal{C})^A$ uses both Ψ and Ψ^{-1}.

Let H be also a bialgebra in $\mathcal{C}$ and $\rho : A \otimes H \to \underline{1}$ a bialgebra pairing.

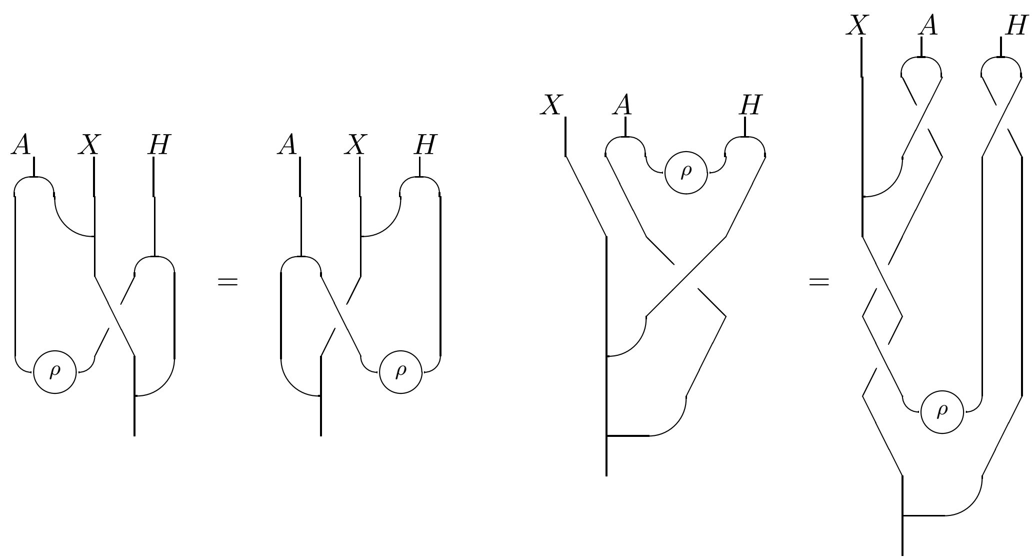
DEFINITION 3.1.2 *Objects of category ${}_A\mathcal{DY}(\mathcal{C})_H$ (resp. $\mathcal{DY}(\mathcal{C})_{A,H}$) are both A- and H- modules satisfying the first (resp. the second) identity in Fig.5b.*

If the pairing is trivial ($\rho = \epsilon_A \otimes \epsilon_H$) then objects of ${}_A\mathcal{DY}(\mathcal{C})_H$ are A-H-bimodules.

Connections between the categories ${}_A^A\mathcal{DY}$, ${}_A\mathcal{DY}^A$, ${}^A\mathcal{DY}_A$, $\mathcal{DY}_A^A$ of crossed modules over an ordinary bialgebra (Hopf algebra) were studied in [34]. Categories $\mathcal{DY}(\mathcal{C})_A^A$ and ${}_A^A\mathcal{DY}(\mathcal{C})$ in the fully braided setting were introduced and studied in the preprint version of the present paper (May 1994). Left-right crossed modules over a braided group were defined independently in [7] where an interesting connection of these objects with topological quantum field theories was discovered. The category $\mathcal{DY}(\mathcal{C})_{A,H}$ turns into the category of modules over the corresponding Drinfel'd's double in the case of ordinary bialgebras A and H.



a) axioms for objects of $\mathcal{DY}(\mathcal{C})_A^A$ and ${}_A\mathcal{DY}(\mathcal{C})^A$



b) axioms for objects of ${}_A\mathcal{DY}(\mathcal{C})_H$ and $\mathcal{DY}(\mathcal{C})_{A,H}$

Figure 5: Drinfel'd-Yetter compatibility conditions.

3.2. We will use abbreviated form $\mathrm{L}^X_{\mathcal{DY}(\mathcal{C})^A_A}$ (resp. $\mathrm{R}^X_{\mathcal{DY}(\mathcal{C})^A_A}$) for the left hand side (resp. for the right hand side) of right crossed module axiom for X over a bialgebra A in $\mathcal{C}$ and similar notations for other variants of crossed modules.

LEMMA 3.2.1 *Let A be a bialgebra in $\mathcal{C}$ and (X, μ_r, Δ_r) right crossed module over A. Then X with opposite action $\mu_r \circ \Psi^{-1}$ is left-right crossed module over A^{op}.*

This construction defines categorical isomorphism $\mathcal{DY}(\mathcal{C})^A_A \simeq {}_{A^{\mathrm{op}}}\mathcal{DY}\left(\overline{\mathcal{C}}\right)^{A^{\mathrm{op}}}$.

Proof.

$$\mathrm{L}^X_{{}_{A^{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}}} = \Psi^{-1}_{X,A} \circ \mathrm{L}^X_{\mathcal{DY}(\mathcal{C})^A_A} = \Psi^{-1}_{X,A} \circ \mathrm{R}^X_{\mathcal{DY}(\mathcal{C})^A_A} = \mathrm{R}^X_{{}_{A^{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}}}$$

□

3.3. Let us describe a monoidal structure on the categories of crossed modules.

LEMMA 3.3.1 *If X and Y are right crossed modules over a bialgebra A then $X \otimes Y$ also is, with module (resp. comodule) structure the braided tensor product one from X and Y defined by the last diagram in Fig.1e. (resp. by the input-output reversed diagram). This turns $\mathcal{DY}(\mathcal{C})^A_A$ into a monoidal category.*

Proof. See Fig.18. □

The following lemma can be verified directly or obtained as a corollary of lemmas 3.2.1 and 3.3.1.

LEMMA 3.3.2 *If X and Y are left-right crossed modules over a bialgebra A then $X \otimes Y$ also is, with underlying module (resp. comodule) the tensor product one in ${}_{A^{\mathrm{op}}}\overline{\mathcal{C}}$ (resp. in $\mathcal{C}^A$).*

We will use modified notation ${}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^A$ for the category of left-right crossed modules with monoidal structure described in previous lemma. An obvious symmetry implies existence of the second monoidal structure ${}_A\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}}$ on this category.

One can also verify that the categories ${}_A\mathcal{DY}(\mathcal{C})_H$, $\mathcal{DY}(\mathcal{C})_{A,H^{\mathrm{op}}}$ are monoidal (where 'op' emphasize that in the second category H-module structure on $X \otimes Y$ is defined by tensor product in $\overline{\mathcal{C}}_{H^{\mathrm{op}}}$).

3.4. (Large) category **MonCat** of all (small) monoidal categories is monoidal with usual Cartesian product of categories and functors. Let Ψ be braiding on monoidal category $\mathcal{C}$. Then tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ together with a natural transformation

$$X \otimes \Psi_{X',Y} \otimes Y' : \ (X \otimes X') \otimes (Y \otimes Y') \to (X \otimes Y) \otimes (X' \otimes Y')$$

define a monoidal functor.

More generally, let $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ be monoidal categories and $F_1 : \mathcal{C}_1 \to \mathcal{C}$, $F_2 : \mathcal{C}_2 \to \mathcal{C}$ monoidal functors (for simplicity, we suppose that $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ are strict monoidal categories and F_1, F_2 are strict monoidal functors). And let $\Psi = \{\Psi_{X,Y} | X \in \mathrm{Obj}(\mathcal{C}_1), Y \in \mathrm{Obj}(\mathcal{C}_2)\}$ be a natural transformation of functors

$$\otimes \circ (F_1 \times F_2) \stackrel{\Psi}{\Longrightarrow} \otimes_{\mathrm{op}} \circ (F_1 \times F_2) : \ \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}\,,$$

where $\otimes$ (resp. $\otimes_{\mathrm{op}}$) is tensor product (resp. opposite tensor product) in $\mathcal{C}$. We consider a pair of functor $F := \otimes \circ (F_2 \times F_1)$ and natural transformation $\lambda : F(\_) \otimes F(\_) \to F(\_ \otimes \_)$, where $\lambda_{Y \times X, Y' \times X'} := \mathrm{id}_{F_2(Y)} \otimes \Psi_{X,Y'} \otimes \mathrm{id}_{F_1(X')}$.

LEMMA 3.4.1 *$(F, \lambda) : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}$ is a monoidal functor iff Ψ satisfies the following 'hexagon identities'*

$$\begin{aligned} \Psi_{X \otimes X', Y} &= (\Psi_{X,Y} \otimes \mathrm{id}_{F_1(X')}) \circ (\mathrm{id}_{F_1(X)} \otimes \Psi_{X',Y})\,, \\ \Psi_{X, Y \otimes Y'} &= (\mathrm{id}_{F_2(Y)} \otimes \Psi_{X,Y'}) \circ (\Psi_{X,Y} \otimes \mathrm{id}_{F_2(Y')})\,, \end{aligned} \tag{3.4.1}$$

for $X, X' \in \mathrm{Obj}(\mathcal{C}_1)$ and $Y, Y' \in \mathrm{Obj}(\mathcal{C}_2)$.

In this case we will say that Ψ is a generalized braiding *for $\mathcal{C}_1 \stackrel{F_1}{\to} \mathcal{C} \stackrel{F_2}{\leftarrow} \mathcal{C}_2$.*

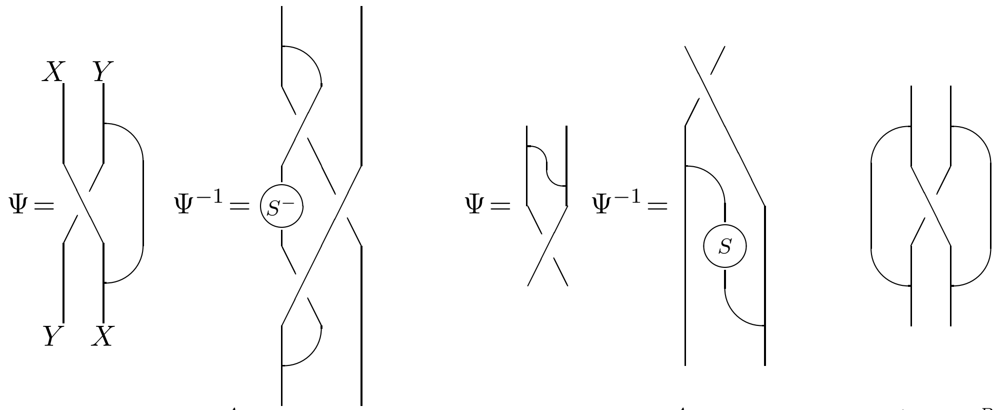
The following is an example a generalized braiding. Let A be a bialgebra in $\mathcal{C}$. For right A-module (X, μ_r^X) and right A-comodule (Y, Δ_r^Y) we define morphism

$$\Psi^A_{X,Y} := (Y \otimes \mu_r^X) \circ (\Psi_{X,Y} \otimes A) \circ (X \otimes \Delta_r^Y) \tag{3.4.2}$$

or, equivalently, by the first diagram in Fig.6.

LEMMA 3.4.2 *The collection $\Psi^A := \{\Psi^A_{X,Y}\}$ is a generalized braiding for pair of underlying functors $\mathcal{C}_A \to \mathcal{C} \leftarrow \mathcal{C}^A$.*

Proof. Fig.19 proves the first identity from (3.4.1). A proof of the second one uses input-output reversed diagrams. □



a) $\mathcal{DY}(\mathcal{C})_A^A$ b) ${}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^A$ c) ${}_A^A\mathcal{DY}(\mathcal{C})_B^B$

Figure 6: Braidings in the categories of crossed modules

THEOREM 3.4.3 *Let A be bialgebra in $\mathcal{C}$. Then the categories $\mathcal{DY}(\mathcal{C})_A^A$ and ${}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^A$ are pre-braided with Ψ shown in Fig.6. If A (resp. A^{op}) is a braided group then ${}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^A$ (resp. $\mathcal{DY}(\mathcal{C})_A^A$) is braided. The lemma 3.2.1 defines isomorphism of (pre-)braided categories $\mathcal{DY}(\mathcal{C})_A^A$ and ${}_{(A^{\mathrm{op}})_{\mathrm{op}}}\mathcal{DY}\left(\overline{\mathcal{C}}\right)^{A_{\mathrm{op}}}$.*

Proof. Hexagon identities for $\mathcal{DY}(\mathcal{C})_A^A$ are proven in the previous lemma. Diagrams in Fig.20 (resp. the input-output reversed diagrams) show that Ψ in Fig.6a is module (resp. comodule) map. It is obvious that Ψ and Ψ^{-1} in Fig.6b are inverse each to other. □

(Pre-)braided structures on ${}_A\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}}$, ${}_A^A\mathcal{DY}(\mathcal{C})$, ${}^A\mathcal{DY}(\mathcal{C})_{A_{\mathrm{op}}}$, ${}^{A^{\mathrm{op}}}\mathcal{DY}(\mathcal{C})_A$ are defined in similar way using the symmetries described in 2.3..

3.5. Let A be a bialgebra (braided group) in $\mathcal{C}$. The lemma 3.2.1 and the following lemmas 3.5.1, 3.5.2, 3.5.6 describe analogues of constructions from 2.5. which produce new crossed modules from given one. As a corollary we obtain isomorphisms between categories of crossed modules.

LEMMA 3.5.1 *Let A be a braided group in $\mathcal{C}$ and (X, μ_ℓ, Δ_r) left-right crossed module over A. Then X with inverse action $\mu_\ell^- := \mu_\ell \circ (S \otimes X)$ is a left-right crossed module over A^{op}.*

This construction extends to braided functor ${}_A\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}} \to {}_{(A^{\mathrm{op}})_{\mathrm{op}}}\mathcal{DY}\left(\overline{\mathcal{C}}\right)^{A^{\mathrm{op}}}$ (isomorphism of braided categories if antipode S is invertible).

Proof. Crossed module axiom for $X^- := (X, \mu_\ell^-, \Delta_r)$ is verified on Fig.21a. $\square$

LEMMA 3.5.2 *Let A be a bialgebra in $\mathcal{C}$ and $X \in \mathrm{Obj}({}^{A}\mathcal{DY}(\mathcal{C})_{A})$, an object $X^\vee$ dual in $\mathcal{C}$ exists and equipped with left A-module and right A-comodule structures defined by the identity in Fig.3c and by the input-output reversed identity. Then $X^\vee \in \mathrm{Obj}({}_{A}\mathcal{DY}(\mathcal{C})^{A})$.*

If $\mathcal{C}$ has (right) duals this construction defines a braided functor from ${}^{A^{\mathrm{op}}}\mathcal{DY}(\mathcal{C})_{A}$ to $({}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^{A})^{\mathrm{op}}_{\mathrm{op}}$.

Proof. Crossed module compatibility condition follows from identities on Fig.21b. $\square$

COROLLARY 3.5.3 *Right dual for object X in the category $\mathcal{DY}(\mathcal{C})_A^A$ of crossed modules over braided group A with invertible antipode is right dual $X^\vee$ for underlying object in $\mathcal{C}$ (if this latter exists) with the action and coaction defined in Fig.3c.*

Proof. Right crossed module structure on $X^\vee$ is the result of sequential application of the constructions from lemma 3.5.2, lemmas 3.2.1, 3.5.1 and their input-output reversed forms. Module (resp. comodule) structure on $X^\vee$ is the same as in the category $\mathcal{C}_A$ (resp. $\mathcal{C}^A$). Hence, pairing and copairing are module and comodule maps. $\square$

LEMMA 3.5.4 *Let A be bialgebra in $\mathcal{C}$. Then ${}_{A}\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}} \stackrel{(\mathrm{Id},\Psi)}{\longrightarrow} ({}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^{A})_{\mathrm{op}}$ is isomorphism of pre-braided categories.*

Taking into attention lemmas 3.2.1, 3.5.1, 3.5.4 and the fact that the functor $\mathcal{C} \stackrel{(\mathrm{Id},\Psi)}{\longrightarrow} \mathcal{C}_{\mathrm{op}}$ defines isomorphism of braided categories, we obtain the following result about isomorphisms between categories of crossed modules as in the case of ordinary Hopf algebras [34].

COROLLARY 3.5.5 *Let A be a braided group in $\mathcal{C}$ with invertible antipode. Then the following braided categories of crossed modules are isomorphic:*

$$\mathcal{DY}(\mathcal{C})_A^A \simeq {}_{A_{\mathrm{op}}}\mathcal{DY}(\mathcal{C})^{A} \simeq {}_{A}\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}} \simeq {}_{A}^{A}\mathcal{DY}(\mathcal{C}) \simeq {}^{A}\mathcal{DY}(\mathcal{C})_{A_{\mathrm{op}}} \simeq {}^{A^{\mathrm{op}}}\mathcal{DY}(\mathcal{C})_{A} .$$

In particular, there exist two braided functors from $\mathcal{DY}(\mathcal{C})_A^A$ to ${}_A^A\mathcal{DY}(\mathcal{C})$ defined on object in the following way:

$$\begin{array}{l}(X,\mu_r,\Delta_r)\mapsto(X,\ \mu_r\circ\Psi^{-1}\circ(S^-\otimes X),\ (S\otimes X)\circ\Psi\circ\Delta_r),\\(X,\mu_r,\Delta_r)\mapsto(X,\ \mu_r\circ\Psi\circ(S\otimes X),\ (S^-\otimes X)\circ\Psi^{-1}\circ\Delta_r).\end{array}\tag{3.5.1}$$

In 3.11. we will describe isomorphism between these functors.

LEMMA 3.5.6 *Let, A and H be bialgebras in $\mathcal{C}$ and $\rho: A\otimes H\to\underline{1}$ a bialgebra pairing. Then any left-right crossed module over A becomes an object of ${}_A\mathcal{DY}(\mathcal{C})_H$ with H-module structure defined via ρ (by input-output reversed form of the last diagram from Fig.3a).*

This construction defines a monoidal functor from ${}_A\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}}$ to ${}_A\mathcal{DY}(\mathcal{C})_H$.

Proof.

$$\mathrm{L}^X_{{}_A\mathcal{DY}(\mathcal{C})_H}=(X\otimes\rho)\circ(\mathrm{L}^X_{{}_A\mathcal{DY}(\mathcal{C})^A}\otimes H)=(X\otimes\rho)\circ(\mathrm{R}^X_{{}_A\mathcal{DY}(\mathcal{C})^A}\otimes H)=\mathrm{R}^X_{{}_A\mathcal{DY}(\mathcal{C})_H}.\qquad\square$$

The image of the functor from the previous lemma is a pre-braided subcategory in ${}_A\mathcal{DY}(\mathcal{C})_H$. In similar way one can define a monoidal functor from $\mathcal{DY}(\mathcal{C})_A^A$ to $\mathcal{DY}(\mathcal{C})_{A,H^{\mathrm{op}}}$.

COROLLARY 3.5.7 *Let A be bialgebra in $\mathcal{C}$ and $A^\vee$ exists. Then the pre-braided categories ${}_A\mathcal{DY}(\mathcal{C})^{A^{\mathrm{op}}}$ and ${}^{(A^\vee)^{\mathrm{op}}}\mathcal{DY}(\mathcal{C})_{A^\vee}$ are isomorphic.*

Finally, we note the following obvious isomorphisms of (pre-)braided categories:

$$\mathcal{DY}(\mathcal{C}^{\mathrm{op}})_A^A\simeq(\mathcal{DY}(\mathcal{C})_A^A)^{\mathrm{op}},\qquad\qquad\mathcal{DY}(\mathcal{C}_{\mathrm{op}})_A^A\simeq({}_A^A\mathcal{DY}(\mathcal{C}))_{\mathrm{op}}.\tag{3.5.2}$$

3.6. Braided category ${}_A^A\mathcal{DY}$ of crossed modules over an ordinary Hopf algebra A can be obtained directly from monoidal category ${}_A\mathcal{M}$ as a 'center' or 'inner double'. This construction is due to Drinfel'd (unpublished) and a formal proof is in [26, 22]. Here we show that the same with slight modification is true in the braided case.

A center $\mathcal{Z}(\mathcal{V})$ of monoidal category $\mathcal{V}$ is a special case of *Pontryagin dual monoidal category* [26, 22]. Objects of $\mathcal{Z}(\mathcal{V})$ are pairs (V,λ_V) where V is an object of $\mathcal{C}$ and λ_V is a natural isomorphism in $\mathrm{Nat}(V\otimes\mathrm{id},\mathrm{id}\otimes V)$ such that

$$\lambda_{V,\underline{1}}=\mathrm{id}\qquad(\mathrm{id}\otimes\lambda_{V,Z})(\lambda_{V,W}\otimes\mathrm{id})=\lambda_{V,W\otimes Z}$$

and morphisms are $\phi:\ V\to W$ such that the objects are intertwined in the form

$$(\mathrm{id}\otimes\phi)\lambda_{V,Z}=\lambda_{W,Z}(\phi\otimes\mathrm{id})\,,\qquad\forall Z\in\mathrm{Obj}(\mathcal{C})$$

The tensor product and braiding are the following:

$$(V,\lambda_V)\otimes(W,\lambda_W)=(V\otimes W,\lambda_{V\otimes W})\qquad \lambda_{V\otimes W,Z}=(\lambda_{V,Z}\otimes \mathrm{id})(\mathrm{id}\otimes\lambda_{W,Z})\,,$$

$$\Psi_{(V,\lambda_V),(W,\lambda_W)}=\lambda_{V,W}$$

Existence of braiding in this case was pointed out by Drinfel'd.

Let A be a braided group with invertible antipode in $\mathcal{C}$. For a special case $\mathcal{V}={}_A\mathcal{C}$ we denote by $\mathcal{Z}_{\mathcal{C}}({}_A\mathcal{C})$ the full subcategory of $\mathcal{Z}({}_A\mathcal{C})$ with the following condition on λ_V: for any object W in $\mathcal{C}$ with trivial action (through counit) $\lambda_{V,W}$ coincides with braiding $\Psi_{V,W}$ in $\mathcal{C}$. (In the tensor category **Vect** of vector spaces the unit object $\underline{1}=k$ is a generator of the category and then $\mathcal{Z}_{\mathrm{Vect}}({}_A\mathcal{M})=\mathcal{Z}({}_A\mathcal{M})$.) The following proposition is analog of corresponding result from [26, 22].

PROPOSITION 3.6.1 *Braided monoidal categories* ${}^A_A\mathcal{DY}(\mathcal{C})$ *and* $\mathcal{Z}_{\mathcal{C}}({}_A\mathcal{C})$ *are isomorphic.*

Proof. Proof is like in the corresponding proposition from [26, 22]. One can identify a crossed module X with a pair of the underlying module of X and the braiding ${}^{({}^A_A\mathcal{DY}(\mathcal{C}))}\Psi_{X,\_}$. Conversely, coaction on X can be reconstructed from (X,λ_X) as the composition morphism $X\xrightarrow{\mathrm{id}\otimes\eta}X\otimes A\xrightarrow{\lambda_X}A\otimes X$. (We consider A with left regular module structure.) $\square$

Similarly, one can identify $\mathcal{DY}(\mathcal{C})^A_A$ and $\mathcal{Z}_{\mathcal{C}}(\mathcal{C}^A)$. So 'centers' of categories of modules and comodules are the same and coincide with the category of crossed modules just as in the standard case over **Vect**.

3.7. In 3.1. we defined the category $\mathcal{DY}(\mathcal{C})_{A,H}$ depending on bialgebra pairing $\rho: A\otimes H\to\underline{1}$, which is a fully braided analog of the category of modules $\mathcal{M}_{\mathcal{D}(A,H,\rho)}$ over Drinfel'd double $\mathcal{D}(A,H,\rho)$ of ordinary Hopf algebras. Here we want to discuss when this category can be realized as a category of modules over something.

We will consider a special case. Let k be a field, $(\Gamma,\cdot)$ an Abelian group, $k\Gamma$ a group algebra equipped with a coquasitriangular structure given by a function $\chi:\Gamma\times\Gamma\to k$ obeying the bicharacter conditions [30]. And let $\mathcal{C}$ be the category $\mathcal{M}^{k\Gamma}$ of right $k\Gamma$-comodules (X,Δ^X_r) (or, equivalently, the category of Γ-graded spaces $X=\oplus X_{\alpha\in\Gamma}$ where $X_\alpha:=\{x\in X|\Delta^X_r(x)=x\otimes\alpha\}$) with braiding defined by this coquasitriangular structure:

$$\Psi: x\otimes y\mapsto \chi(\alpha,\beta)\cdot y\otimes x\,,\quad x\in X_\alpha\,,\ y\in Y_\beta\,.$$

Let $A=\oplus_{\alpha\in\Gamma}A_\alpha$, $H=\oplus_{\alpha\in\Gamma}H_\alpha$ be braided groups and $\rho: A\otimes H\to k$ a bialgebra pairing in $\mathcal{C}$ (where $\otimes$ means a tensor product over k). We denote by G a copy of

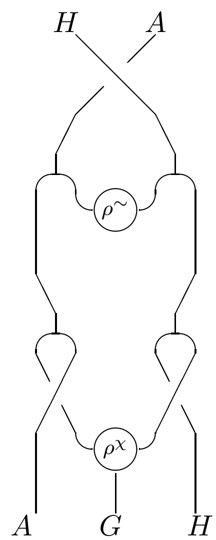


Figure 7: Multiplication in $A \otimes G \otimes H$.

$k\Gamma$ considered as an object of $\mathcal{C}$ with the trivial $k\Gamma$-comodule structure (really, G is a Hopf algebra in $\mathcal{C}$). $k\Gamma$-comodule property of ρ means that $\rho(a \otimes h) \neq 0$ for $a \in A_\alpha, h \in H_\beta$ only if $\alpha = \beta^{-1}$. The formula $\rho^\chi(a \otimes h) := \rho(a \otimes h) \cdot \alpha$ for such a and h define 'G-valued pairing' $\rho^\chi : A \otimes H \to G$ in $\mathcal{C}$. (Note that the underlying algebra of G is commutative and there are no problems to transform definition of bialgebra pairing from 2.4. to this case.) Let us also define right G-module structure on X in $\mathcal{C}$ by the formula

$$X_\alpha \otimes G \ni x_\alpha \otimes \beta \mapsto \chi(\alpha, \beta) \cdot \chi(\beta, \alpha) \cdot x_\alpha \in X_\alpha \,. \tag{3.7.1}$$

There exists an algebra structure on $A \otimes G \otimes H$ which is uniquely determined by the following conditions that A, G, H are subalgebras of $A \otimes G \otimes H$ with embeddings $A \otimes \eta_G \otimes \eta_H$, $\eta_A \otimes G \otimes \eta_H$ and $\eta_A \otimes \eta_G \otimes H$ respectively, G is in the center of $A \otimes G \otimes H$, the product $a \cdot \alpha \cdot h$ for $a \in A$, $\alpha \in G$, $h \in H$ equals to $a \otimes \alpha \otimes h$, and the product $h \cdot a := (1_A \otimes 1_G \otimes h) \cdot (a \otimes 1_G \otimes 1_H)$ is defined by the diagram in Fig.7, where crossings mean inverse to braiding in $\mathcal{C}$.

PROPOSITION 3.7.1 *The category $\mathcal{DY}(\mathcal{C})_{A,H}$ is identified with a full subcategory of $\mathcal{C}_{(A \otimes G \otimes H)}$ of $(A \otimes G \otimes H)$-modules such that underlying G-module structure coincides with (3.7.1).*

Let us consider an example. Let $\mathcal{C}$ be a category of $\mathbb{Z}$-graded vector spaces $X = \oplus_{n \in \mathbb{Z}} X_n$ with braiding

$$x \otimes y \mapsto q^{mn} \cdot y \otimes x \qquad \text{for} \quad x \in X_m, \;\; y \in Y_n, \tag{3.7.2}$$

where $q \in k^*$ is a fixed invertible element such that $q^n \neq 1$ for $n = 1, 2, \ldots$. Majid's braided line is the simplest example of a properly braided group. This is a free k-algebra $A := k[x]$ with one generator x of degree 1. Counit, comultiplications and antipode are the following:

$$\epsilon(x^n) = \delta_{n,0}\,, \qquad \Delta(x^n) = \Sigma_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \otimes x^{n-k}\,, \qquad S(x^n) = (-1)^n \cdot q^{\frac{n(n-1)}{2}} \cdot x^n\,, \tag{3.7.3}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}\,, \qquad [n]_q! := \Pi_{k=1}^{n} [k]_q\,, \qquad [n] := \frac{q^n - 1}{q - 1}\,.$$

We also consider a similar Hopf algebra H in $\mathcal{C}$ with one generator y of degree -1. Majid noted that there exists the following non-degenerated bialgebra pairing between A and H:

$$\rho : A \otimes H \to k\,, \qquad x^m \otimes y^n \mapsto \delta_{m,n} \cdot [n]_q!\,. \tag{3.7.4}$$

We want to describe the corresponding category $\mathcal{DY}(\mathcal{C})_{A,H}$. Let $G := k[t, t^{-1}]$ be a group algebra of free group with one generator t. We consider G as an object of $\mathcal{C}$ and suppose that degree of t is 0. The formula in Fig.7 for multiplication in $A \otimes G \otimes H$ takes the form

$$y^n \cdot x^m = \sum_{\substack{k,\ell,m',n' \geq 0 \\ m'+k+\ell = m \\ n'+k+\ell = n}} (-1)^k \cdot q^{mn + [\frac{k}{2}] - \ell(m'+n')} \cdot \frac{[m]_q! \cdot [n]_q!}{[m']_q! \cdot [n']_q! \cdot [k]_q! \cdot [\ell]_q!} \cdot x^{m'} \otimes t^{\ell} \otimes y^{n'}\,. \tag{3.7.5}$$

In particular, $yx = q(xy + t - 1)$.

Let us consider a quiver with points labeled by integers and arrows x_n, y_n, $n \in \mathbb{Z}$:

$$\cdots \ \circ_{n-1} \ \underset{y_n}{\overset{x_{n-1}}{\rightleftarrows}} \ \circ_n \ \underset{y_{n+1}}{\overset{x_n}{\rightleftarrows}} \ \circ_{n+1} \ \cdots$$

And let $\mathcal{A}$ be the category generated by this quiver with relations $y_{n+1} \cdot x_n = q \cdot x_{n-1} \cdot y_n + q(q^{2n} - 1) \cdot 1_n$.

PROPOSITION 3.7.2 *The category $\mathcal{DY}(\mathcal{C})_{A,H}$ is identified with the category of representations of $\mathcal{A}$ over* **Vect***.*

*Representation $\pi : \mathcal{A} \to$ **Vect** corresponds to crossed module from $\mathcal{DY}(\mathcal{C})_A^A \hookrightarrow \mathcal{DY}(\mathcal{C})_{A,H}$ (where embedding is described by the lemma 3.5.6) iff there exists $n_0 \in \mathbb{Z}$ such that $\pi(n)$ is a zero vector space for all $n < n_0$. In particular, representations of $\mathcal{A}$ with vacuum state correspond to simple A-crossed modules.*

One can also consider the category of $\mathbb{Z}_n$-graded (anionic) spaces with braiding defined by the formula (3.7.2) with $q^n = 1$ and algebras $A = k[x]/(x^n)$, $H = k[y]/(y^n)$ in this category with braided group structure defined by (3.7.3). A similar approach allows us to describe simple crossed modules over A. This give us examples of anyonic R-matrix, in particular, those described in [31].

More detailedly these results will be published anywhere.

3.8. For a pair of braided groups (bialgebras) A and B in $\mathcal{C}$ one can consider a category ${}^A_A\mathcal{DY}(\mathcal{C})^B_B$ whose objects X are both left crossed modules over A with action μ_ℓ and coaction Δ_ℓ and right crossed modules over B with action μ_r and coaction Δ_r such that (X, μ_ℓ, μ_r) is A-B-bimodule, $(X, \Delta_\ell, \Delta_r)$ is A-B-bicomodule and

$$\Delta_r \circ \mu_\ell = (\mu_\ell \otimes B) \circ (A \otimes \Delta_r), \qquad \Delta_\ell \circ \mu_r = (A \otimes \mu_r) \circ (\Delta_\ell \otimes B).$$

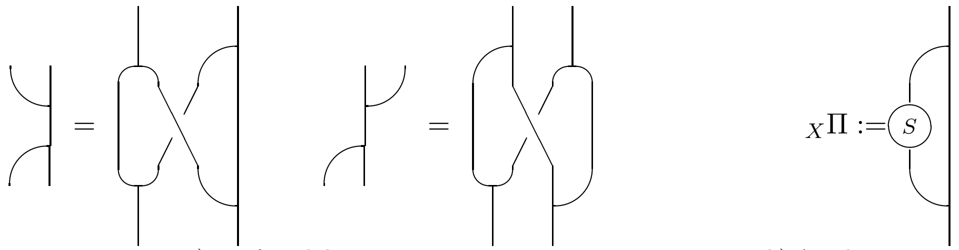
${}^A_A\mathcal{DY}(\mathcal{C})^B_B$ is a monoidal category with standard tensor product for underlying (co)modules and can be equipped with (pre-)braiding defined in Fig.6c. This and more complicated structures will be studied in forthcoming papers.

3.9. A Hopf bimodules appeared (under the name 'bicovariant bimodules') as the basic notion in Woronowicz approach to differential calculus on quantum groups [39]. The main theorem of the paper [36] can be considered as a coordinate free version of the Woronowicz result about structure of Hopf bimodules and states the equivalence between (pre-)braided categories of Hopf bimodules and crossed modules over a Hopf algebra in a symmetric monoidal category which has (co)equalizers. In [5] the same result is proven in the context of braided groups without assumption about existence of (co)equalizers. Consideration of Hopf bimodules simplifies proofs of certain results about crossed bimodules. Here we describe necessary facts from [5].

Firstly, we need an assumption that idempotents are split in our braided category $\mathcal{C}$, i.e. for any idempotent (projection) $e = e^2 : X \to X$ there exist object X_e and morphisms $X_e \underset{p_e}{\overset{i_e}{\rightleftarrows}} X$ such that $e = i_e \circ p_e$ and $p_e \circ i_e = \mathrm{id}_{X_e}$. This condition can be always satisfied:

LEMMA 3.9.1 *For any braided category $\mathcal{C}$ there exists braided category $\widetilde{\mathcal{C}}$ and full embedding of braided categories $\mathcal{C} \hookrightarrow \widetilde{\mathcal{C}}$ with idempotents in $\widetilde{\mathcal{C}}$ split.*

Proof. We need only to prove that standard construction of idempotent splitting is compatible with braided structure. Objects in the category $\widetilde{\mathcal{C}}$ are pairs $X_e = (X, e)$,



a) Hopf module axioms b) An idempotent ${}_X\Pi$

Figure 8:

where X is object in $\mathcal{C}$ and $e: X \to X$ is idempotent: $e^2 = e$. Morphisms are the following

$$\widetilde{\mathcal{C}}(X_e, Y_f) := \{t \in \mathcal{C}(X, Y) | fte = t\}$$

with ordinary composition. Tensor product and braiding are:

$$X_e \otimes Y_f := (X \otimes Y)_{e \otimes f}$$

$$\Psi_{X_e, Y_f} := (f \otimes e) \circ \Psi_{X,Y} \circ (e \otimes f) = (f \otimes e) \circ \Psi_{X,Y} = \Psi_{X,Y} \circ (e \otimes f)$$

The axioms of braided category are easily verified. One can identify $\mathcal{C}$ with the full subcategory of $\widetilde{\mathcal{C}}$ whose objects are (X, id_X). $\square$

We note that if idempotents are split in $\mathcal{C}$ then the same is true in the categories $\mathcal{C}_A$, $\mathcal{C}^A$ and $\mathcal{DY}(\mathcal{C})_A^A$.

DEFINITION 3.9.1 A left Hopf module *(resp.* right-left Hopf module*) X over a bialgebra A is a left (resp. right) A-module and left A-comodule with the compatibility condition (Fig.8a) which is a "polarized" version of the bialgebra axiom (Fig.1c).*

A Hopf bimodule $X = (X, \mu_\ell, \mu_r, \Delta_\ell, \Delta_r)$ *over a bialgebra A is an object X which is A-A-bimodule and A-A-bicomodule and such that any pair of action and coaction defines on X structure of a Hopf module.*

Hopf bimodules together with the A-bimodule-A-bicomodule morphisms form the category which will be denoted by ${}^A_A\mathcal{C}^A_A$

LEMMA 3.9.2 *Let X be a Hopf bimodule over a braided group A. Then the following are right crossed module structures on its underlying object: X_{ad} with adjoint action and regular coaction, X^{ad} with regular action and adjoint coaction.*

Morphism ${}_X\Pi$ defined in Fig.8b is an idempotent and ${}_X\Pi : X^{\mathrm{ad}} \to X_{\mathrm{ad}}$ is a crossed module map.

One can define a crossed module ${}_AX$ (*the object of left invariants*) and crossed module morphisms ${}_Xp : X^{\mathrm{ad}} \to {}_AX$ and ${}_Xi : {}_AX \to X_{\mathrm{ad}}$ which split ${}_X\Pi$, i.e. ${}_X\Pi = {}_Xi \circ {}_Xp$.

LEMMA 3.9.3 *Let $X = (X, \mu_\ell, \Delta_\ell, \mu_r, \Delta_r)$ be a Hopf bimodule and Y a right crossed module over a bialgebra A. Then $X \otimes Y$ equipped with right (co)module structure tensor product one from X and Y and left action $\mu_\ell \otimes Y$ and coaction $\Delta_\ell \otimes Y$ is a Hopf bimodule.*

In the case when $X = A$ is a regular Hopf bimodule we use notation $A \ltimes Y$ for tensor product equipped with a Hopf module structure described in the previous lemma.

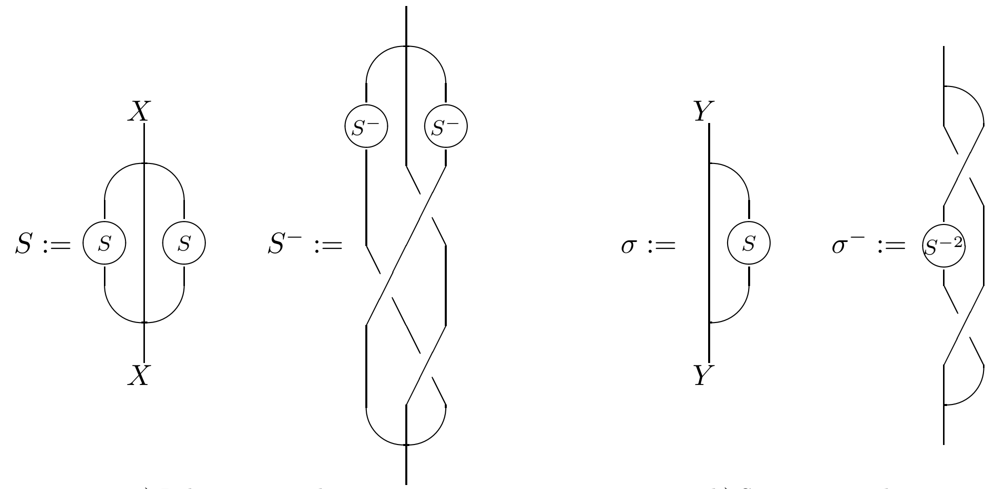
PROPOSITION 3.9.4 *Let A be a braided group in $\mathcal{C}$. Then the constructions described above extend to functors*

$$ {}^A_A\mathcal{C}^A_A \underset{A\ltimes(\_)}{\overset{{}_A(\_)}{\rightleftarrows}} \mathcal{DY}(\mathcal{C})^A_A \tag{3.9.1}$$

We define the following tensor products on the category ${}^A_A\mathcal{C}^A_A$. For any Hopf bimodules $(X, \mu_\ell, \Delta_\ell, \mu_r, \Delta_r)$ and $(X', \mu'_\ell, \Delta'_\ell, \mu'_r, \Delta'_r)$ we denote by $X \overset{\circ}{\otimes} X'$ (resp. by $X \underset{\circ}{\otimes} X'$) tensor product of their underlying left and right modules (resp. comodules) equipped with left and right comodule structures $\Delta_\ell \otimes X'$, $X \otimes \Delta'_r$ (resp. module structures $\mu_\ell \otimes X'$, $X \otimes \mu'_r$) and put $f \overset{\circ}{\otimes} f' := f \otimes f' =: f \underset{\circ}{\otimes} f'$ for Hopf bimodule maps f and f'. One can directly verify that $\overset{\circ}{\otimes}$ and $\underset{\circ}{\otimes}$ are bifunctors ${}^A_A\mathcal{C}^A_A \times {}^A_A\mathcal{C}^A_A \to {}^A_A\mathcal{C}^A_A$ satisfying the associativity condition. But there are no units for these tensor products.

THEOREM 3.9.5 *Let A be a braided group in $\mathcal{C}$. Then there exist pre-braided structure $({}^A_A\mathcal{C}^A_A, \underset{A}{\otimes}, A, \Psi = {}^{({}^A_A\mathcal{C}^A_A)}\Psi)$ (braided if antipode S_A is invertible) and natural transformation of functors $(\_) \underset{\circ}{\otimes} (\_) \xrightarrow{\phi^{\otimes}_{\_,\_A}} (\_) \underset{A}{\otimes} (\_) \xrightarrow{\phi^{\square}_{\_,\_A}} (\_) \overset{\circ}{\otimes} (\_)$ such that*

- *any $\phi^{\otimes}_{X,X'}$ (resp. $\phi^{\square}_{X,X'}$) is a (co)tensor product over A, i.e. coequalizer of $X \otimes A \otimes X' \underset{X\otimes\mu'_\ell}{\overset{\mu_r\otimes X'}{\rightrightarrows}} X \otimes X'$ (resp. equalizer of $X \otimes X' \underset{X\otimes\Delta'_\ell}{\overset{\Delta_r\otimes X'}{\rightrightarrows}} X \otimes A \otimes X'$);*
- *functors (3.9.1) define equivalence of (pre-)braided categories.*



a) Relative antipode b) Square antipode

Figure 9:

3.10. Important examples are crossed modules A_{ad}, A^{ad} related with the regular Hopf bimodule over a braided group A. New solutions of Yang–Baxter equation arise as braiding between these objects in the category of crossed modules [31]. Connection of A_{ad} with bicovariant differential calculi on A was pointed out (in coordinate form) by Woronowicz [39] (see also [5] for fully braided setting).

A new concept of *a braided Lie algebra* was introduced and studied by Majid [24]. He showed that A_{ad} satisfy the axioms of braided Lie algebra if underlying module of A_{ad} is an object of $\mathcal{C}_{\mathcal{O}(A)}$. He also built enveloping bialgebra $U(\mathcal{L})$ for any braided Lie algebra $\mathcal{L}$ and defined adjoint action of $U(\mathcal{L})$ on $\mathcal{L}$ and on itself. We would like to note here that $U(\mathcal{L})_{\mathrm{ad}}$ is a crossed module over $U(\mathcal{L})$ with crossed submodule $\mathcal{L}$.

3.11. As was noted in [2], one can define analog of antipode (square of antipode) for any Hopf bimodule (resp. crossed module). See [5] for more details about relative antipode.

DEFINITION 3.11.1 *Let A be a braided group, X a Hopf bimodule and Y a right crossed module over A.* (Relative) antipode $S = S_{X/A}$ *for X (resp.* square antipode $\sigma = \sigma_{Y/A}$ *for Y) is defined by the first formula on Fig.9a (resp. in Fig.9b), where* 3*-vertices denote compositions of left and right (co)actions.*

For the regular Hopf bimodule relative antipode coincide with the usual Hopf algebra antipode. The polarized forms of the two last identities in Fig.1d are true:

$$S_{X/A}\circ\mu_r=\mu_\ell\circ\Psi\circ(S_{X/A}\otimes S_A)\qquad\text{(and three similar identities).}\tag{3.11.1}$$

If antipode of a braided group is invertible then relative antipode for a Hopf bimodule also has inverse given by the second formula in Fig.9. This follows directly from (3.11.1).

Idempotent Π_X for a Hopf bimodule X over a braided group A is defined by left-right reversed form of the diagram in Fig.8. Relative antipode transforms idempotents Π_X and ${}_X\Pi$ one to other:

$$S_{X/A}\circ\Pi_X={}_X\Pi\circ\Pi_X={}_X\Pi\circ S_{X/A},\qquad S_{X/A}\circ{}_X\Pi=\Pi_X\circ{}_X\Pi=\Pi_X\circ S_{X/A}\tag{3.11.2}$$

COROLLARY 3.11.1 *For a Hopf bimodule X and for the corresponding right crossed module ${}_AX$ the following identities are true:*

$$S^2_{X/A}\circ{}_X\Pi={}_X\Pi\circ S^2_{X/A}\circ{}_X\Pi={}_X\Pi\circ S^2_{X/A}\,,\tag{3.11.3}$$

$${}_Xi\circ\sigma_{{}_AX/A}=S^2_{X/A}\circ{}_Xi\,,\qquad\sigma_{{}_AX/A}\circ{}_Xp={}_Xp\circ S^2_{X/A}\,.\tag{3.11.4}$$

COROLLARY 3.11.2 *Let A be a braided group with invertible antipode and (Y,μ_r^Y,Δ_r^Y) a right crossed module over A. Then*

$$(Y,\,\mu_r^Y\circ\Psi^{-1}\circ(S^-\otimes Y),\,(S\otimes Y)\circ\Psi\circ\Delta_r^Y)\xrightarrow{\sigma_{Y/A}}(Y,\,\mu_r^Y\circ\Psi\circ(S\otimes Y),\,(S^-\otimes Y)\circ\Psi^{-1}\circ\Delta_r^Y)$$

is a morphism of left crossed modules. Moreover a collection of $\sigma_{Y/A}$ describe isomorphism between two corresponding braided functors from $\mathcal{DY}(\mathcal{C})_A^A$ to ${}_A^A\mathcal{DY}(\mathcal{C})$ defined in 3.5..

Morphism $\sigma^-_{Y/A}$ defined in Fig.9b is inverse to $\sigma_{Y/A}$.

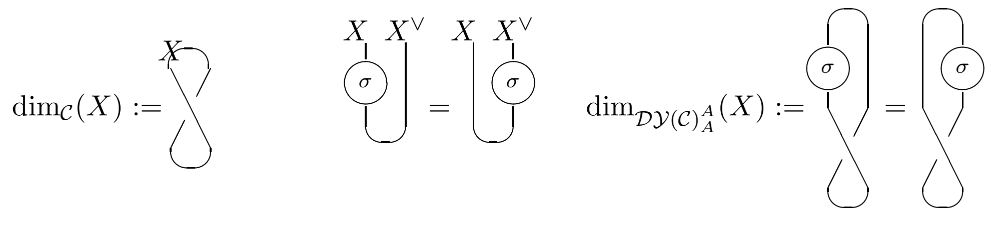
Proof. Let us put $X=A\ltimes Y$. Then one can identify Y with ${}_AX$. Formulas (3.11.1) imply that

$$S^2_{X/A}\circ\mu_r^X\circ\Psi^{-1}\circ(S_A^-\otimes{}_AX)=\mu_r^X\circ\Psi\circ(S_A\otimes S^2_{X/A})\,.$$

Composition with ${}_Xi$ and ${}_Xp$ and then application of (3.11.4) prove that $\sigma_{{}_AX/A}$ is a modules morphism. One can also find a direct proof of the (co-)module property of $\sigma_{X/A}$ in the preprint version of this paper.

Module or comodule property of $\sigma_{Y/A}$ imply that both $\sigma_{Y/A}\circ\sigma^-_{Y/A}$ and $\sigma^-_{Y/A}\circ\sigma_{Y/A}$ are equal to

$$\mu_r^Y\circ(\mu_r^Y\otimes A)\circ(Y\otimes S_A\otimes A)\circ(\Delta_r^Y\otimes A)\circ\Delta_r^Y=\mathrm{id}_Y\,.\qquad\square$$



a) A rank of $X \in \mathrm{Obj}(\mathcal{C})$ b) A rank of $X \in \mathrm{Obj}(\mathcal{DY}(\mathcal{C})_A^A)$

Figure 10:

For right crossed modules A^{ad} and A_{ad} we have $\sigma_{A^{\mathrm{ad}}/A} = \sigma_{A_{\mathrm{ad}}/A} = S_A^2$ as in the case of ordinary Hopf algebra [25].

PROPOSITION 3.11.3 *The following identity is true for any crossed modules X and Y from $\mathcal{DY}(\mathcal{C})_A^A$:*

$$^{(\mathcal{DY}(\mathcal{C})_A^A)}\Psi_{Y,X} \circ \sigma_{(X\otimes Y)/A} \circ {}^{(\mathcal{DY}(\mathcal{C})_A^A)}\Psi_{X,Y} = {}^{\mathcal{C}}\Psi_{Y,X} \circ (\sigma_{Y/A} \otimes \sigma_{X/A}) \circ {}^{\mathcal{C}}\Psi_{X,Y} \quad (3.11.5)$$

Proof. See Fig.22. □

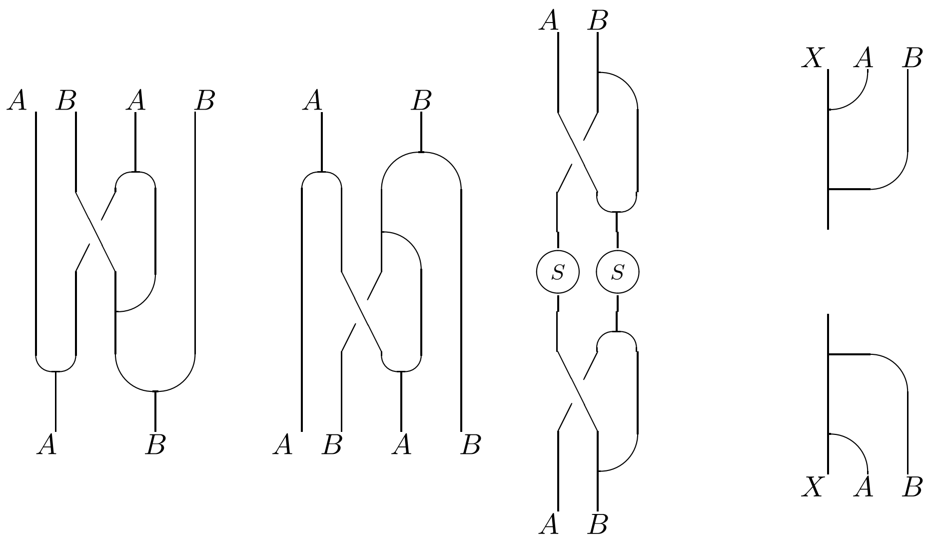
Square antipode σ naturally appears in the formula for rank of crossed module. Let $X \in \mathrm{Obj}(\mathcal{C})$ and $X^\vee$ exists. *A rank* or *categorical dimension* [28] of X is the element of $\mathrm{End}(\underline{1})$ defined on Fig.10a.

PROPOSITION 3.11.4 *Let X be a crossed module with dual $X^\vee$. Then the identities in Fig.10b are true.*

4 Cross products.

4.1. A structure of cross product algebra can be realized as a tensor product algebra in a suitable braided category.

PROPOSITION 4.1.1 *(Cf. [31]) Let A be a Hopf algebra in a braided category $\mathcal{C}$. Then A_{ad} (resp. A^{ad}) become commutative algebra (resp. cocommutative coalgebra) in the category $\mathcal{DY}(\mathcal{C})_A^A$ with multiplication (resp. comultiplication) inherited from A.*



a) Braided group structure. b) (Co)action.

Figure 11: Cross product $A \ltimes B$.

Let B be a Hopf algebra in a category $\mathcal{DY}(\mathcal{C})^A_A$. Denote by $A \ltimes B$ object $A \otimes B$ equipped with algebra structure $A_{\mathrm{ad}} \otimes B$, coalgebra structure $A^{\mathrm{ad}} \otimes B$ (tensor product algebra and coalgebra in $\mathcal{DY}(\mathcal{C})^A_A$) and antipode

$$S_{A \ltimes B} := {}^{(\mathcal{DY}(\mathcal{C})^A_A)}\Psi_{B,A_{\mathrm{ad}}} \circ (S_B \otimes S_A) \circ {}^{(\mathcal{DY}(\mathcal{C})^A_A)}\Psi_{A^{\mathrm{ad}},B}.$$

(It's easy to see that there exists $S_{A \ltimes B}{}^{-}$ if S_A^{-} and S_B^{-} exist because all factors in the last formula are invertible.) Formulas for multiplication, comultiplication and antipode in terms of the category $\mathcal{C}$ are given in Fig.11a.

THEOREM 4.1.2 *$A \ltimes B$ is a Hopf algebra in a category $\mathcal{C}$.*

Proof. Underlying algebra of $A \ltimes B$ is exactly Majid's cross product by braided group in [27] so associativity is clear. The coalgebra is precisely dual construction by input-output symmetry. Nontrivial part is a verification of the bialgebra axiom (see Fig.23). □

This is the analog of construction of such cross products and cross coproducts for ordinary Hopf algebras in [32], [17]. The analog of the converse, which is the Majid-Radford theorem [32], [23] is also true in our braided situation.

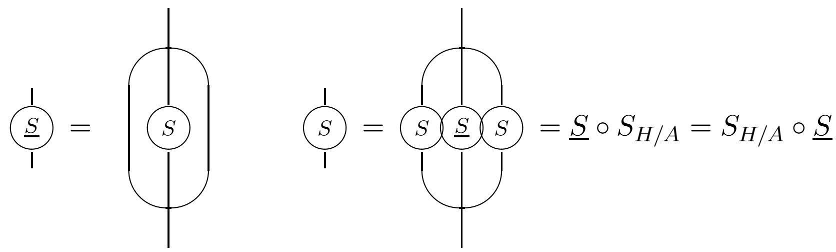


Figure 12: Relations between antipodes on H and $\underline{H}$.

Let A and H be bialgebras in $\mathcal{C}$. A bialgebra projection is a pair of bialgebra morphisms

$$A \overset{i_A}{\to} H \overset{p_A}{\to} A \tag{4.1.1}$$

such that the composition homomorphism $p_A \circ i_A$ equals id_A. A direct proof of the following theorem was given in the preprint version of this paper.

THEOREM 4.1.3 *Let $\mathcal{C}$ be a braided category with split idempotents, A and H braided groups in $\mathcal{C}$, and bialgebra projection (4.1.1) be given. Then there exists a braided group B living in the category $\mathcal{DY}(\mathcal{C})_A^A$ such that $H \simeq A \ltimes B$.*

4.2. For given a bialgebra projection (4.1.1) one can turn H into an object $\underline{H}$ of ${}_A^A\mathcal{C}_A^A$ taking composition of the regular actions (resp. coactions) with i_A (resp. p_A). Both theorems 4.1.2 and 4.1.3 are corollaries of the following theorem and the result about equivalence of the categories ${}_A^A\mathcal{C}_A^A$ and $\mathcal{DY}(\mathcal{C})_A^A$.

THEOREM 4.2.1 *(cf. [5]) Let $(A, \mu_A, \eta_A, \Delta_A, \epsilon_A)$ be a braided group in $\mathcal{C}$. For any bialgebra $(H, \mu_H, \eta_H, \Delta_H, \epsilon_H)$ and bialgebra projection (4.1.1) morphism μ (resp. Δ) is uniquely factorized through (co-)tensor product over A:*

$$\mu_H = \underline{\mu} \circ \phi^{\otimes_A}_{\underline{H},\underline{H}}, \qquad \Delta_H = \phi^{\Box_A}_{\underline{H},\underline{H}} \circ \underline{\Delta}, \tag{4.2.1}$$

And $\underline{H} := (H, \underline{\mu}, \underline{\eta}, \underline{\Delta}, \underline{\epsilon})$ is a bialgebra in ${}_A^A\mathcal{C}_A^A$, where $\underline{\eta} := i_A$ and $\underline{\epsilon} := p_A$. It this way one to one correspondence between bialgebra projections (4.1.1) and bialgebras in ${}_A^A\mathcal{C}_A^A$ is defined.

There also exists one to one correspondence between braided group structures on H and $\underline{H}$. The relations between corresponding antipodes S and $\underline{S}$ are shown in Fig.12.

The idempotent ${}_H\Pi$ for $H \in \mathrm{Obj}({}_A^A\mathcal{C}_A^A)$ from the previous theorem takes the form

$${}_H\Pi = \mu_H \circ (i_A \circ S_A \circ p_A \otimes H) \circ \Delta_H\,. \tag{4.2.2}$$

Let $B \underset{p_B}{\overset{i_B}{\rightleftarrows}} H$ split this idempotent.

COROLLARY 4.2.2 *B is a bialgebra in the category $\mathcal{DY}(\mathcal{C})_A^A$ with multiplication $p_B \circ \mu_H \circ (i_B \otimes i_B)$, comultiplication $(p_B \otimes p_B) \circ \Delta_H \circ i_B$, right A-module structure $p_B \circ \mu_H \circ (i_B \otimes i_A)$ and right A-comodule structure $(p_B \otimes p_A) \circ \Delta_H \circ i_B$. This bialgebra satisfy the theorem 4.1.3.*

Let A be a braided group in a category $\mathcal{C}$, B a Hopf algebra in $\mathcal{DY}(\mathcal{C})_A^A$ and X be a right module over B in $\mathcal{DY}(\mathcal{C})_A^A$. It is easy to verify that the first diagram in Fig.11b defines $(A \ltimes B)$-module structure on X. Moreover, in this way one can construct full embedding of categories $(\mathcal{DY}(\mathcal{C})_A^A)_B \hookrightarrow \mathcal{C}_{A \ltimes B}$

PROPOSITION 4.2.3 *Let A be a braided group in $\mathcal{C}$ and B a braided group in $\mathcal{DY}(\mathcal{C})_A^A$. Then the braided categories $(\mathcal{DY}(\mathcal{C})_A^A)_B^B$ and $\mathcal{DY}(\mathcal{C})_{A \ltimes B}^{A \ltimes B}$ are isomorphic.*

Proof. $(A \ltimes B)$-(co-)module structure on object X of $\mathcal{DY}\left(\mathcal{DY}(\mathcal{C})_A^A\right)_B^B$ is defined in Fig.11b. Nontrivial part is a verification of crossed module axiom for X in Fig.24. Conversely, A-, (B-) crossed module structure on objects of $\mathcal{DY}(\mathcal{C})_{A \ltimes B}^{A \ltimes B}$ is defined by means of i_A, p_A (i_B, p_B). $\square$

So for object C in $\mathcal{C}$ to be a braided group in $\mathcal{DY}(\mathcal{C})_{A \ltimes B}^{A \ltimes B}$ or in $\mathcal{DY}\left(\mathcal{DY}(\mathcal{C})_A^A\right)_B^B$ are equivalent. In this case on $A \otimes B \otimes C$ there exist two braided group structures $(A \ltimes B) \ltimes C$ and $A \ltimes (B \ltimes C)$.

PROPOSITION 4.2.4 (Transitivity of cross product) *Braided groups $(A \ltimes B) \ltimes C$ and $A \ltimes (B \ltimes C)$ coincide.*

4.3. The following results about cross products involve a square antipode $\sigma_{X/A}$.

LEMMA 4.3.1 *Let $H = A \ltimes B$ be a cross product of braided groups A in $\mathcal{C}$ and B in $\mathcal{DY}(\mathcal{C})_A^A$. Then for any object X of $\mathcal{DY}(\mathcal{C})_H^H = \mathcal{DY}\left(\mathcal{DY}(\mathcal{C})_A^A\right)_B^B$*

$$\sigma_{X/H} = \sigma_{X/A} \circ \sigma_{X/B} = \sigma_{X/B} \circ \sigma_{X/A}. \tag{4.3.1}$$

LEMMA 4.3.2 *Square of antipode in a cross product $A \ltimes B$ of braided groups has the form*

$$S_{A \ltimes B}{}^2 = (S_A^2 \otimes S_B^2) \circ \Psi^2 \circ (A \otimes \sigma_{B/A}). \tag{4.3.2}$$

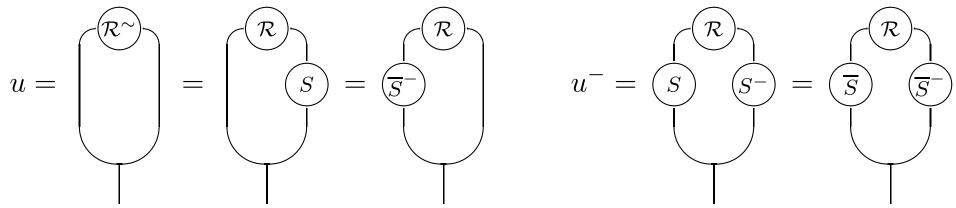


Figure 13:

5 Remarks on Quantum braided groups.

5.1. We note that in definition of quasitriangular bialgebra $(A, \overline{A}, \mathcal{R})$ coassociativity and bialgebra axiom for $\overline{A}$ follow from other axioms.

Really, the bialgebra axiom is versified directly (taking into attention that $\mathcal{R}^-$ is a copairing between ${\overline{A}_{\mathrm{op}}}^{\mathrm{op}}$ and A^{op}):

$$\overline{\Delta}^{\mathrm{op}} \circ \mu_A = \mathcal{R} \cdot (\Delta \circ \mu_A) \cdot \mathcal{R}^- = \mathcal{R} \cdot (\mu_{A\otimes A} \circ (\Delta \otimes \Delta)) \cdot \mathcal{R}^- =$$

$$= \mu_{A\otimes A} \circ (\mathcal{R} \cdot \Delta \cdot \mathcal{R}^- \otimes \mathcal{R} \cdot \Delta \cdot \mathcal{R}^-) = \mu_{A\otimes A} \circ (\overline{\Delta}^{\mathrm{op}} \otimes \overline{\Delta}^{rmop})$$

Then one can use axioms of quasitriangular structure, coassociativity of Δ, bialgebra axiom for A and $\overline{A}$ to prove coassociativity of $\overline{\Delta}^{\mathrm{op}}$:

$$(\overline{\Delta}^{\mathrm{op}} \otimes A) \circ \overline{\Delta}^{\mathrm{op}} = \mathcal{R}_{23} \cdot \mathcal{R}_{13} \cdot \mathcal{R}_{12} \cdot ((\Delta \otimes \mathrm{id}) \circ \Delta) \cdot \mathcal{R}^-_{12} \cdot \mathcal{R}^-_{13} \cdot \mathcal{R}^-_{23} = (A \otimes \overline{\Delta}^{\mathrm{op}}) \circ \overline{\Delta}^{\mathrm{op}}\,.$$

The following obvious lemma is important for our further considerations.

LEMMA 5.1.1 *If $(A, \overline{A}, \mathcal{R})$ is a quantum braided group (quasitriangular bialgebra) in $\mathcal{C}$ then $(\overline{A}, A, \overline{\mathcal{R}})$ is a quantum braided group (quasitriangular bialgebra) in $\overline{\mathcal{C}}$. (Pre-)braided categories $\overline{\mathcal{C}}_{\mathcal{O}(\overline{A},A)}$ and $\overline{\mathcal{C}_{\mathcal{O}(A,\overline{A})}}$ are naturally identified.*

5.2. Here we describe relations between antipodes in quantum braided group and distinguished 'element' u. These are analogues of the results obtained in [9], [35], [33] for ordinary quantum groups.

Let $(A, \overline{A}, \mathcal{R})$ be a quantum braided group and $X \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}(A,\overline{A})})$. Then $\sigma_{X^{\mathcal{R}}/A}$ is a result of action $\triangleleft u$, where $u = \mu_A \circ \mathcal{R}^\sim$ (see Fig.13). Denote by $\overline{u}$ corresponding elements for quantum braided group $(\overline{A}, A, \overline{\mathcal{R}})$.

PROPOSITION 5.2.1 *'Elements' $u, \overline{u}$ have inverse $u^-, (\overline{u})^-$ (where u^- is defined in Fig.13). Antipodes in quantum braided group are invertible. The following relations are true:*

$$\overline{S}^- = u \cdot S \cdot u^-, \qquad S^- = \overline{u} \cdot \overline{S} \cdot (\overline{u})^-\,,$$

$$\overline{u} = S^{-} \circ u^{-} = \overline{S} \circ u^{-}\,, \qquad \overline{u}^{-} = S^{-} \circ u = \overline{S} \circ u\,,$$

where $u \cdot S$ is abbreviated notation for $\mu \circ (u \otimes S)$, etc.

The following (braided variant of the formula for Δu) is true for any modules X and Y from $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$:

$$^{(\mathcal{C}_{\mathcal{O}(H)})}\Psi_{Y,X} \circ (\triangleleft u) \circ {}^{(\mathcal{C}_{\mathcal{O}(H)})}\Psi_{X,Y} = {}^{\mathcal{C}}\Psi_{Y,X} \circ ((\triangleleft u) \otimes (\triangleleft u)) \circ {}^{\mathcal{C}}\Psi_{X,Y} \qquad (5.2.1)$$

Proof. The axiom of quasitriangular structure in Fig.4a implies directly that

$$\mu \circ (A \otimes u \cdot S) \circ \overline{\Delta}^{\mathrm{op}} = u \circ \epsilon\,, \qquad \mu \circ (S \cdot u^{-} \otimes A) \circ \overline{\Delta}^{\mathrm{op}} = u^{-} \circ \epsilon \qquad (5.2.2)$$

with $u^{-} := \mu \circ (S \circ \overline{S} \otimes A) \circ \mathcal{R}$ Application of $\overline{S}$ to the first identity from (5.2.2) and then convolution product with id_A give: $u \cdot (S \circ \overline{S}) = \mathrm{id}_A \cdot u$. The later identity implies that $u^{-} := \mu \circ (S \circ \overline{S} \otimes A) \circ \mathcal{R}$ is right inverse to u. Then it follows from (5.2.2) that $u \cdot S \cdot u^{-}$ is a skew antipode for $\overline{A}$. This and the same considerations for $(\overline{A}, A, \overline{\mathcal{R}})$ prove the first part of the theorem.

The second part is a special case of (3.11.5). □

For the ordinary quantum group our $(\overline{u})^{-}$ coincide with u from [9], [35].

5.3. Let $(A, \overline{A}, \mathcal{R})$ be a quasitriangular bialgebra in $\mathcal{C}$. It is convenient to describe the category $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ in terms of crossed modules. For right A-module X denote by $X^{\mathcal{R}}$ object with additional comodule structure defined by the last diagram in Fig.3a. An idea to turn a module into a comodule by $\mathcal{R}$ is due to Majid [17]. A proof of the following lemma is obtained immediately in a similar way from axioms of quasitriangular structure.

LEMMA 5.3.1 *Let Y be a right module over A. Then* $Y \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}(A,\overline{A})})$ *iff* $Y^{\mathcal{R}} \in \mathrm{Obj}\left(\mathcal{DY}\left(\mathcal{C}\right)_A^A\right)$.

In this case $(X \otimes Y)^{\mathcal{R}} = X^{\mathcal{R}} \otimes Y^{\mathcal{R}}$ for any module X.

So one can identify $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ with a full braided monoidal subcategory of $\mathcal{DY}\left(\mathcal{C}\right)_H^H$ and braiding coincides with that defined by Majid (Fig.4c).

LEMMA 5.3.2 *Let $(A, \overline{A}, \mathcal{R})$ be a quantum braided group in $\mathcal{C}$ and* $X \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}(A,\overline{A})})$. *Then the identities in Fig.14 are true. (In the second case we suppose that there exist right dual $X^{\vee}$ in $\mathcal{C}$.)*

Proof. Both parts of the first (resp. the second) identity on Fig.14 equals the first (resp. the second) diagram in Fig.25 . □

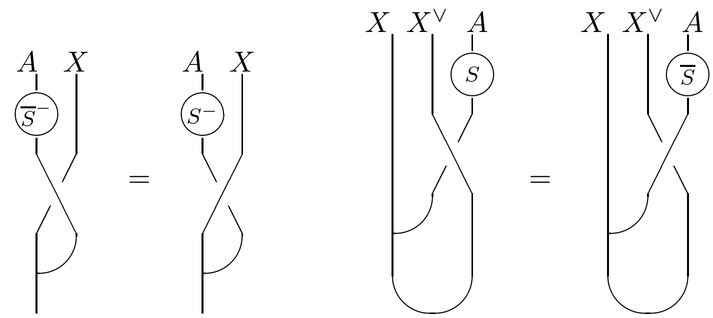


Figure 14:

COROLLARY 5.3.3 *Let X be a right module from $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ and there exists left dual ${}^\vee X$ (resp. right dual $X^\vee$) then $({}^\vee X)^{\mathcal{R}} = {}^\vee(X^{\mathcal{R}})$ (resp. $(X^\vee)^{\mathcal{R}} = (X^{\mathcal{R}})^\vee$)* and ${}^\vee X$ *(resp. $X^\vee$) belongs to $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$.*

Proof. Proposition is a direct corollary of the previous lemma and the dual form of identities (2.4.2) $(A \otimes S^{\pm 1}) \circ \mathcal{R} = (\overline{S}^{\mp 1} \otimes A) \circ \mathcal{R}$. For example, proof for the left dual is in Fig.26. $\square$

5.4. Obviously, one can consider left-right reversed form of quantum braided group (quantum braided group in $\mathcal{C}_{\mathrm{op}}$), corresponding subcategory of left modules and embedding of this category into the category of left crossed modules. Let $(A, \overline{A}, \mathcal{R})$ be a quantum braided group in $\mathcal{C}$. Then $(A_{\mathrm{op}}, \overline{A}_{\mathrm{op}}, \overline{R})$ is a quantum braided group in $\mathcal{C}_{\mathrm{op}}$. Then, it follows from 3.5. that we can turn any object $\left(X, \mu_\ell, \Delta_\ell := (A \otimes \mu_\ell) \circ (\overline{\mathcal{R}} \otimes X)\right)$ of $\overline{\mathcal{C}}_{\mathcal{O}(A,\overline{A})} \subset {}^{A_{\mathrm{op}}}_{A_{\mathrm{op}}}\mathcal{DY}\left(\overline{\mathcal{C}}\right)$ into an object of ${}^A_A\mathcal{DY}(\mathcal{C})$ with modified coaction $\Psi \circ (X \otimes S) \circ \Psi \circ \Delta_\ell$, and this construction is extended to isomorphism of braided categories. We show in Fig.15 comodule structure on an object of ${}_{\mathcal{O}(\overline{A}_{\mathrm{op}}, A_{\mathrm{op}})}\overline{\mathcal{C}}$ and braiding on ${}_{\mathcal{O}(\overline{A}_{\mathrm{op}}, A_{\mathrm{op}})}\overline{\mathcal{C}}$ [27] obtained in this way.

All our further results about bosonization and transmutation have analogues for left modules from ${}_{\mathcal{O}(\overline{A}_{\mathrm{op}}, A_{\mathrm{op}})}\overline{\mathcal{C}}$. But formulas in terms of initial quantum braided group $(A, \overline{A}, \mathcal{R})$ for the last case are more complicated (corresponding diagrams have more crossings). So we prefer to work with right modules.

6 Generalized bosonization and transmutation.

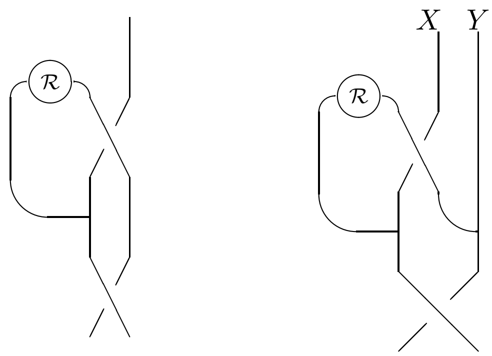


Figure 15:

6.1. Here we give cross product construction for quantum groups in braided categories which generalize Majid's bosonization construction $A \ltimes B$ to the case where A is a quantum braided group rather than a quantum group as in [27]. The usual bosonization assigns this ordinary Hopf algebra to any braided-Hopf algebra in the category of A-modules. Majid also showed in [27] that when B is braided quasitriangular then $A \ltimes B$ has a 'relative quasitriangular structure' in the sense of a second coproduct and an $\mathcal{R}_{A\ltimes B}$. This construction extends straightforwardly to our setting where A is itself a quantum braided group. We also extend the braided version of Majid-Radford theorem in section 4 to the case where the braided groups are equipped with quasitriangular structures respected by the projections.

Let $(A, \overline{A}, \mathcal{R}_A)$ be a quantum braided group in $\mathcal{C}$ and $(B, \overline{B}, \mathcal{R}_B)$ a quantum braided group in $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. Then one can use embeddings $\mathcal{C}_{\mathcal{O}(A,\overline{A})} \to \mathcal{DY}(\mathcal{C})_A^A$ to construct cross-product braided group $H := A \ltimes B$ in $\mathcal{C}$. Combining formulas for comultiplication from Fig.11a and for comodule structure from Fig.3a one can obtain an expression for comultiplication in $A \ltimes B$ (Fig.16). Braided group $\overline{H} := \overline{A} \ltimes \overline{B}$ in $\overline{\mathcal{C}}$ is built in the same way. Let us define a quasitriangular structure $\mathcal{R}_H$ as a result of multiplication applied to $\mathcal{R}_A$ and $\mathcal{R}_B$ as shown in Fig.16 (we consider A and B as subalgebras of H). Written explicitly this is exactly the form [29] of a cross coproduct by the induced coaction as for the usual bosonization theorem.

THEOREM 6.1.1 $(A \ltimes B, \overline{A} \ltimes \overline{B}, \mathcal{R}_{A\ltimes B})$ *is a quantum braided group in* $\mathcal{C}$.
Braided categories $\mathcal{C}_{\mathcal{O}(A\ltimes B, \overline{A}\ltimes\overline{B})}$ *and* $(\mathcal{C}_{\mathcal{O}(A,\overline{A})})_{\mathcal{O}(B,\overline{B})}$ *are isomorphic.*

Proof. 'Elements' $\mathcal{R}_{A\ltimes B}$ and $\overline{\mathcal{R}}^{\mathrm{op}}_{A\ltimes B} := \Psi \circ \mathcal{R}_{\overline{A}\ltimes\overline{B}}$ are inverse to each other:

$$\overline{\mathcal{R}}^{\mathrm{op}}_{A\ltimes B} \cdot \mathcal{R}_{A\ltimes B} = \ldots \quad \text{See Fig.27} \quad \ldots = \eta_{A\ltimes B} \otimes \eta_{A\ltimes B} \,.$$

It is shown in Fig.28a) that $\mathcal{R}_H$ is an algebra-coalgebra copairing. After series of transformations one can verify that both $\overline{\Delta}^{\mathrm{op}}_{A\ltimes B} \cdot \mathcal{R}_{A\ltimes B}$ and $\mathcal{R}_{A\ltimes B} \cdot \Delta_{A\ltimes B}$

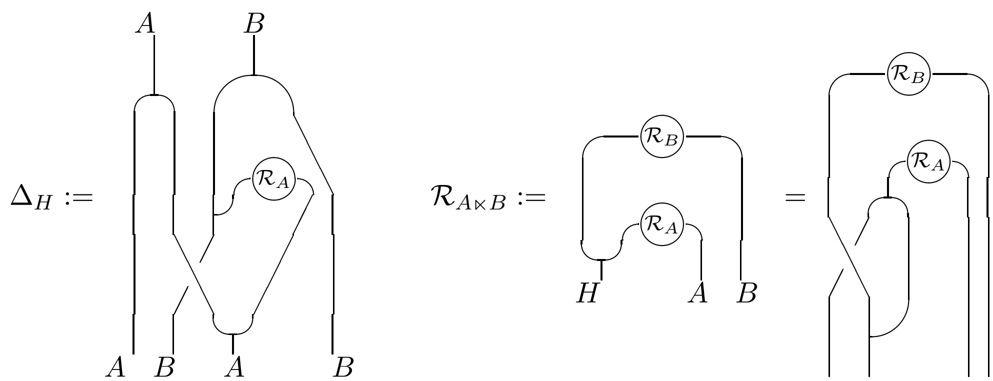


Figure 16: Quantum braided group structure on cross product $A \ltimes B$

are equal to morphism given by the diagram in Fig.28c) (everywhere dot means multiplication in braided tensor product algebra in corresponding category $\mathcal{C}$ or $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$).

For any bialgebra A in $\mathcal{C}$ and right A-module (X, Δ_r^X) let us temporarily denote by $\psi^{\mathcal{C}}_{X,A}$ the composition morphism $(A \otimes \Delta_r^X) \circ {}^{\mathcal{C}}\Psi_{X,A} \circ (X \otimes \Delta_A)$ To proof the second part of the theorem we note that in our case for any object X of the category $\mathcal{C}_{A\ltimes B} = (\mathcal{C}_A)_B$ the following identities are true:

$$\begin{aligned} \psi^{\mathcal{C}}_{X,A\ltimes B} &= (A \otimes \psi^{\mathcal{C}_{\mathcal{O}(A,\overline{A})}}_{X,B}) \circ (\psi^{\mathcal{C}}_{X,A} \otimes B)\,, \\ \psi^{\overline{\mathcal{C}}}_{X,\overline{A}\ltimes\overline{B}} &= (A \otimes \psi^{\overline{\mathcal{C}}_{\mathcal{O}(\overline{A},A)}}_{X,B}) \circ (\psi^{\overline{\mathcal{C}}}_{X,\overline{A}} \otimes B)\,. \end{aligned} \qquad (6.1.1)$$

But condition that $X \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}(H,\overline{H})})$ (resp. $X \in \mathrm{Obj}((\mathcal{C}_{\mathcal{O}(A,\overline{A})})_{\mathcal{O}(B,\overline{B})})$) is exactly coincidence of the left hand sides (resp. the right hand sides) of (6.1.1). $\square$

As in the case of braided groups (theorem 4.2.4) cross product of quantum braided groups is transitive:

THEOREM 6.1.2 *Let* $(A, \overline{A}, \mathcal{R}_A)$ *be a quantum group in* $\mathcal{C}$, $(B, \overline{B}, \mathcal{R}_B)$ *a quantum group in* $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$, $(C, \overline{C}, \mathcal{R}_C)$ *a quantum group in* $\mathcal{C}_{\mathcal{O}(A\ltimes B,\overline{A}\ltimes\overline{B})} = (\mathcal{C}_{\mathcal{O}(A,\overline{A})})_{\mathcal{O}(B,\overline{B})}$. *Then quantum groups* $\Big((A \ltimes B) \ltimes C, (\overline{A} \ltimes \overline{B}) \ltimes \overline{C}, \mathcal{R}_{(A\ltimes B)\ltimes C}\Big)$ *and* $\Big(A \ltimes (B \ltimes C), \overline{A} \ltimes (\overline{B} \ltimes \overline{C}), \mathcal{R}_{A\ltimes(B\ltimes C)}\Big)$ *coincide.*

Proof. It is a corollary of the theorem 4.2.4. One needs only to verify that $\mathcal{R}_{(A\ltimes B)\ltimes C} = \mathcal{R}_{A\ltimes(B\ltimes C)}$ using associativity of multiplication in $A \ltimes B \ltimes C$. $\square$

DEFINITION 6.1.1 *Let* $(A, \overline{A}, \mathcal{R}_A)$ *and* $(H, \overline{H}, \mathcal{R}_H)$ *be quantum groups in* $\mathcal{C}$*. Pair of morphisms* $A \underset{p_A}{\overset{i_A}{\rightleftarrows}} H$ *is called* a quantum group projection *if:*

- *both* $A \underset{p_A}{\overset{i_A}{\rightleftarrows}} H$ *and* $\overline{A} \underset{p_A}{\overset{i_A}{\rightleftarrows}} \overline{H}$ *are bialgebra projections in* $\mathcal{C}$ *and* $\overline{\mathcal{C}}$ *respectively.*
- $${}_H\Pi = {}_{\overline{H}}\Pi\,,$$ *where the first (resp. the second) is an idempotent for* $H \in \mathrm{Obj}({}^A_A\mathcal{C}^A_A)$ *defined by (4.2.2) (resp. corresponding idempotent for* $\overline{H} \in \mathrm{Obj}({}^{\overline{A}}_{\overline{A}}\overline{\mathcal{C}}^{\overline{A}}_{\overline{A}})$*) (For ordinary quantum groups* A *and* H *this condition is satisfied automatically.);*
- *and*

$$(H \otimes p_A) \circ \mathcal{R}_H = (i_A \otimes A) \circ \mathcal{R}_A\,, \qquad (p_A \otimes H) \circ \mathcal{R}_H = (A \otimes i_A) \circ \mathcal{R}_A\,. \tag{6.1.2}$$

THEOREM 6.1.3 *Let* $\mathcal{C}$ *be a braided category with split idempotents and* $(A, \overline{A}, \mathcal{R}_A) \underset{p_A}{\overset{i_A}{\rightleftarrows}} (H, \overline{H}, \mathcal{R}_H)$ *a quantum group projection in* $\mathcal{C}$*. Then there exists quantum group* $(B, \overline{B}, \mathcal{R}_B)$ *in the category* $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ *such that*

$$(H, \overline{H}, \mathcal{R}_H) \simeq (A \ltimes B, \overline{A} \ltimes \overline{B}, \mathcal{R}_{A \ltimes B})\,. \tag{6.1.3}$$

Proof. Let $H \underset{i_B}{\overset{p_B}{\rightleftarrows}} B$ split idempotent ${}_H\Pi$ in $\mathcal{C}$. According to theorem 4.1.3 and corollary 4.2.2 one can equip B with a structure of Hopf algebra in $\mathcal{DY}(\mathcal{C})^A_A$ and with a structure of Hopf algebra $\overline{B}$ in $\mathcal{DY}\left(\overline{\mathcal{C}}\right)^{\overline{A}}_{\overline{A}}$ with the same underline algebra. Canonical Hopf algebra isomorphisms $H \simeq A \ltimes B$ and $\overline{H} \simeq \overline{A} \ltimes \overline{B}$ are defined. And we will identify H with $A \ltimes B$.

The axiom of quasitriangular structure (Fig.4a) for $\mathcal{R}_H$ implies that

$$(p_B \otimes p_A) \circ (\mathcal{R}_H \cdot \Delta_H) \circ i_B = (p_B \otimes p_A) \circ (\overline{\Delta}^{\mathrm{op}}_H \cdot \mathcal{R}_H) \circ i_B \tag{6.1.4}$$

One can use sequentially the fact that $p_A : H \to A$ is an algebra morphism, identities (6.1.2) and $p_B \circ \mu_H \circ (i_H \otimes H) = \epsilon_A \otimes p_B$ (or, respectively the dual form $(p_A \otimes H) \circ \overline{\Delta}_H \circ i_B = \eta_A \otimes i_B$) to show that the left hand side of (6.1.4) equals to $(p_B \otimes p_B) \circ \Delta_H \circ i_B$ (that is a right A-comodule structure on B) and, respectively, the right hand side of (6.1.4) equals to $(p_B \circ \mu_H \circ (i_B \otimes i_A) \otimes A) \circ (B \otimes \mathcal{R}_A)$ (that

is the composition of a right A-module structure on B with the quasitriangular structure $\mathcal{R}_A$). And so B is an object of the subcategory $\mathcal{C}_{\mathcal{O}(A,\overline{A})} \subset \mathcal{DY}(\mathcal{C})_A^A$.

Let us define

$$\mathcal{R}_B := (p_B \otimes p_B) \circ \mathcal{R}_H \,. \tag{6.1.5}$$

One can see that

$$\begin{aligned}(i_B \otimes B) \circ \mathcal{R}_B &= (\Pi_B \otimes p_B) \circ \mathcal{R}_H \\ &= (H \otimes p_B) \circ \Big(((i_A \circ \overline{S}_A \circ p_A \otimes H) \circ \mathcal{R}_H) \cdot \mathcal{R}_H \Big) \\ &= (H \otimes p_B) \circ \Big(((i_A \circ \overline{S}_A \otimes i_A) \circ \mathcal{R}_A) \cdot \mathcal{R}_H \Big) \\ &= (H \otimes p_B) \circ \mathcal{R}_H \,. \end{aligned} \tag{6.1.6}$$

And taking into account this identity we obtain the expression for $\mathcal{R}_H$ via $\mathcal{R}_A$ and $\mathcal{R}_B$ as in Fig.16:

$$\mathcal{R}_H = (H \otimes (p_A \otimes p_B) \circ \Delta_H) \circ \mathcal{R}_H = (H \otimes p_A \otimes p_B) \circ ((\mathcal{R}_H)_{13} \cdot (\mathcal{R}_H)_{23}) = (\mathcal{R}_B)_{13} \cdot (\mathcal{R}_A)_{23}$$

The identity

$$(p_B \otimes p_B) \circ (\mathcal{R}_H \cdot \Delta_H) \circ i_A = (p_B \otimes p_B) \circ (\overline{\Delta}_H^{\mathrm{op}} \cdot \mathcal{R}_H) \circ i_A \tag{6.1.7}$$

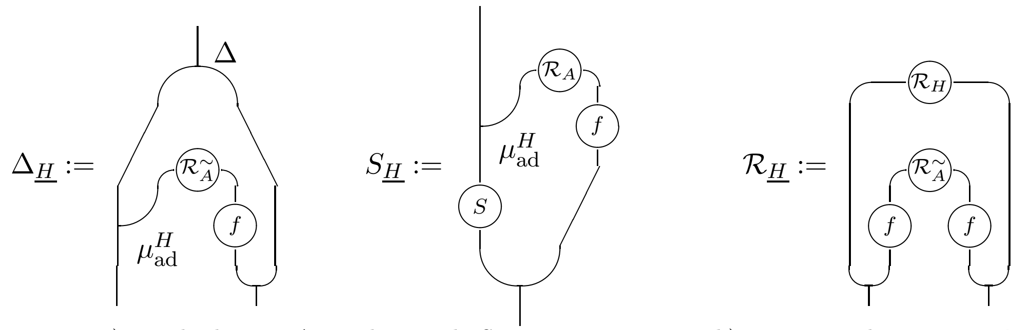
imply that $\mathcal{R}_B$ is A-module morphism. Compositions with $p_B^{\otimes 2}$ or $p_B^{\otimes 3}$ turn axioms of quasitriangular structure for $\mathcal{R}_H$ into corresponding axioms for $\mathcal{R}_B$. $\square$

6.2. A notion of a transmutation, a basic theory and non-trivial examples of braided groups obtained via transmutation belong to Majid [19, 28, 27]. We describe here a braided version of Majid's transmutation and show that analogues of Majid results (in particular, transitivity of transmutation) keep in this situation. Transmutation of a crossed module is defined and compatibility of transmutation with cross product of braided groups is proven. In particular, connection between transmutation and bosonization noted in [27] is generalized for the fully braided setting.

DEFINITION 6.2.1 *Let $(A, \overline{A}, \mathcal{R}_A)$ be a quantum braided group in $\mathcal{C}$, (Y, μ_ℓ^Y, μ_r^Y) bimodule over its underlying algebra. We say that $\mu_{\mathrm{ad}}^Y : Y \otimes A \to Y$ is* a (right) adjoint action of quantum braided group $(A, \overline{A}, \mathcal{R}_A)$ on bimodule Y *if μ_{ad}^Y is adjoint action on Y for both bialgebras A and $\overline{A}$.*

We denote by ${}_A\mathcal{C}_{\mathcal{O}(A,\overline{A})}$ a full subcategory of the category ${}_A\mathcal{C}_A$ of bimodules Y over A such that there exists adjoint action μ_{ad}^Y of $(A, \overline{A}, \mathcal{R}_A)$ on Y and $(Y, \mu_{\mathrm{ad}}^Y) \in$ Obj$(\mathcal{C}_{\mathcal{O}(A,\overline{A})})$. *Let, moreover, $(X, \mu_r^X) \in$* Obj$(\mathcal{C}_{\mathcal{O}(A,\overline{A})})$. *We define morphism*

$$\tau_{X,Y} := (\mu_r^X \otimes \mu_r^Y) \circ (X \otimes \mathcal{R}_A^{\sim} \otimes Y) : \; X \otimes Y \to X \otimes Y \,. \tag{6.2.1}$$



a) comultiplication $\Delta_{\underline{H}}$ and antipode $S_{\underline{H}}$ b) quasitriangular structure $\mathcal{R}_{\underline{H}}$

Figure 17: Transmutation $\underline{H}$ of (quantum) braided group H.

DEFINITION 6.2.2 *Let $(A, \overline{A}, \mathcal{R}_A)$ be a quantum braided group and (H, μ_H, Δ_H) a bialgebra (braided group) in $\mathcal{C}$ and $f : A \to H$ a bialgebra morphism. Then H becomes a bimodule over A with actions $\mu_\ell^H := \mu_H \circ (f \otimes H)$, and $\mu_r^H := \mu_H \circ (H \otimes f)$. We say that $\Big((A, \overline{A}, \mathcal{R}_A), f, H\Big)$ are* transmutation data for a bialgebra (braided group) H in $\mathcal{C}$ *if (H, μ_ℓ^H, μ_r^H) is an object of ${}_A\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. Let us denote by μ_{ad}^H adjoint action of A on H.* Transmutation $\underline{H}$ of bialgebra (braided group) H *is the underlying algebra of H with new comultiplication $\Delta_{\underline{H}} := \tau_{H,H} \circ \Delta_H$ (and antipode $S_{\underline{H}}$) defined on Fig.17a.*

We denote by $\mathcal{DY}(\mathcal{C})_{H,\mathcal{O}(A,\overline{A})}^H$ full subcategory of $\mathcal{DY}(\mathcal{C})_H^H$ with objects (X, μ_r^X, Δ_r^X) such that X with A-module structure $\mu_r^X \circ (X \otimes f)$ belongs to $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. A transmutation $\underline{X}$ *of a such crossed module is the underline module of X with a new 'coaction' $\Delta_r^{\underline{X}} := \tau_{X,H} \circ \Delta_r^X$.*

THEOREM 6.2.1 *Transmutation $\underline{H}$ of a bialgebra (resp. braided group) H in $\mathcal{C}$ is a bialgebra (resp. braided group) in $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$.*

A full embedding of (pre-)braided categories $\mathcal{DY}(\mathcal{C})_{H,\mathcal{O}(A,\overline{A})}^H \to \mathcal{DY}\Big(\mathcal{C}_{\mathcal{O}(A,\overline{A})}\Big)_{\underline{H}}^{\underline{H}}$ is defined by assignment $X \mapsto \underline{X}$ on objects and identity on morphisms.

Proof. It follows directly from the definition of adjoint action that (H, μ_{ad}^H) becomes an algebra and (X, μ_r^X) a right a H-module in $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$.

The next important step is to show that $\Delta_r^{\underline{X}}$ is A-module morphism. A proof use the definition of adjoint action of quantum braided group $(A, \overline{A}, \mathcal{R}_A)$ on H and the fact that X is an object of both categories $\mathcal{DY}(\mathcal{C})_H^H$ and $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. Application of this result to crossed module H_{ad} implies that $\Delta_{\underline{H}}$ is also A-module morphism.

A-module property of $\Delta_{\underline{H}}$ (resp. of $\Delta_r^{\underline{X}}$), bialgebra axiom for H and the fact that $\mathcal{R}_A$ is a bialgebra copairing imply the coassociativity of $\Delta_{\underline{H}}$ (resp. $\underline{H}$-comodule axiom for $\underline{X}$).

The bialgebra axiom for $\underline{H}$ is a corollary of the following more general result (in the unbraided form noted by Majid): Let B be a right H-module algebra such that its underlying object with right A-module structure defined via f is an object of $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. Then $\tau_{B,H} : B \otimes H \to \underline{B} \otimes \underline{H}$ is an algebra isomorphism, where the first (resp. the second) is the tensor product algebra in $\mathcal{C}$ (resp. in $\mathcal{C}_{A,\overline{A}}$).

The antipode axiom $\mu_H \circ (S_{\underline{H}} \otimes H) \circ \Delta_{\underline{H}} = \eta_H \circ \epsilon_H$ follows directly from the definition. To prove that $\mu_H \circ (H \otimes S_{\underline{H}}) \circ \Delta_{\underline{H}} = \eta_H \circ \epsilon_H$ one needs to rewrite the left hand side of this identity in the form $\mu_r^H \circ (\ldots \circ \mu_r^H \otimes A) \circ (H \otimes \mathcal{R}_A)$ using A-module property of $\Delta_{\underline{H}}$ and the fact that $\mathcal{R}_A$ is a coalgebra-algebra copairing. The antipode axiom for $S_{\underline{H}}$ and A-module properties of $\Delta_{\underline{H}}$ and μ_H imply that $S_{\underline{H}}$ is A-module morphism also.

The following identities prove the crossed module axiom for $\underline{X}$:

$$\mathrm{L}^{\underline{X}}_{\mathcal{DY}\left(\mathcal{C}_{\mathcal{O}(A,\overline{A})}\right)^{\underline{H}}_{\underline{H}}} = \tau_{X,H} \circ \mathrm{L}^X_{\mathcal{DY}(\mathcal{C})^H_H} = \tau_{X,H} \circ \mathrm{R}^X_{\mathcal{DY}(\mathcal{C})^H_H} = \mathrm{R}^{\underline{X}}_{\mathcal{DY}\left(\mathcal{C}_{\mathcal{O}(A,\overline{A})}\right)^{\underline{H}}_{\underline{H}}}. \qquad \square$$

DEFINITION 6.2.3 *Let* $(A, \overline{A}, \mathcal{R}_A)$ *be a quantum braided group and* $(H, \overline{H}, \mathcal{R}_H)$ *a quasitriangular bialgebra (resp. a quantum braided group) in* $\mathcal{C}$*. We say that a morphism* $f : A \to H$ *defines* transmutation data for a quasitriangular bialgebra (resp. for a quantum braided group) $(H, \overline{H}, \mathcal{R}_H)$ *if* $\Big((A, \overline{A}, \mathcal{R}_A), f, H\Big)$ *are transmutation data for a bialgebra (braided group)* H *in* $\mathcal{C}$ *and* $\Big((\overline{A}, A, \overline{\mathcal{R}_A}), f, \overline{H}\Big)$ *are transmutation data for a bialgebra (braided group)* $\overline{H}$ *in* $\overline{\mathcal{C}}$*. In this case we define* a transmutation of $(H, \overline{H}, \mathcal{R}_H)$ *as a triple which consists of* $\underline{H}$ *(transmutation of* H*),* $\underline{\overline{H}}$ *(transmutation of* $\overline{H}$*) and* $\mathcal{R}_{\underline{H}}$ *(transmutation of quasitriangular structure defined in Fig.17b).*

THEOREM 6.2.2 $\left(\underline{H}, \underline{\overline{H}}, \mathcal{R}_{\underline{H}}\right)$ *is a quasitriangular bialgebra (resp. a quantum braided group) in* $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$.

We have the following commutative diagram of full embeddings of (pre-)braided categories

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{O}(H,\overline{H})} & \longrightarrow & \left(\mathcal{C}_{\mathcal{O}(A,\overline{A})}\right)_{\mathcal{O}(\underline{H},\underline{\overline{H}})} \\ \downarrow & & \downarrow \\ \mathcal{DY}(\mathcal{C})^H_{H,\mathcal{O}(A,\overline{A})} & \longrightarrow & \mathcal{DY}\left(\mathcal{C}_{\mathcal{O}(A,\overline{A})}\right)^{\underline{H}}_{\underline{H}} \end{array} \qquad (6.2.2)$$

THEOREM 6.2.3 *(Transitivity of transmutation)*
Let $(A_i, \overline{A}_i, \mathcal{R}_i)$ $i = 0, 1, 2$ *be quantum braided groups in* $\mathcal{C}$ *and morphisms* $f : A_0 \to A_1$, $g : A_1 \to A_2$ *be such that the following triples are transmutation data:*

$$\Big((A_0, \overline{A}_0, \mathcal{R}_0),\ f,\ (A_1, \overline{A}_1, \mathcal{R}_1)\Big)\ , \qquad \Big((A_0, \overline{A}_0, \mathcal{R}_0),\ g \circ f,\ (A_2, \overline{A}_2, \mathcal{R}_2)\Big)$$

and let $(\underline{A_i}, \underline{\overline{A}_i}, \underline{\mathcal{R}_i})$ $i = 1, 2$ *be corresponding transmutations which are quantum braided groups in* $\mathcal{C}_{\mathcal{O}(A_0, \overline{A}_0)}$*. Then* $\Big((\underline{A_1}, \underline{\overline{A}_1}, \underline{\mathcal{R}_1}),\ g,\ (\underline{A_2}, \underline{\overline{A}_2}, \underline{\mathcal{R}_2})\Big)$ *are transmutation data iff* $\Big((A_1, \overline{A}_1, \mathcal{R}_1),\ g,\ (A_2, \overline{A}_2, \mathcal{R}_2)\Big)$ *are the same. In this case corresponding transmutations are quantum braided groups in* $(\mathcal{C}_{\mathcal{O}(A_0, \overline{A}_0)})_{\mathcal{O}(\underline{A_1}, \underline{\overline{A}_1})}$ *and in* $\mathcal{C}_{\mathcal{O}(A_1, \overline{A}_1)}$ *respectively and are the same if we identify* $\mathcal{C}_{\mathcal{O}(A_1, \overline{A}_1)}$ *with a full subcategory of* $(\mathcal{C}_{\mathcal{O}(A_0, \overline{A}_0)})_{\mathcal{O}(\underline{A_1}, \underline{\overline{A}_1})}$*.*

A transmutation is compatible with a braided group cross product and generalized bosonization:

THEOREM 6.2.4 *Let* $\Big((A, \overline{A}, \mathcal{R}_A), f, H\Big)$ *be transmutation data for bialgebra (braided group)* H *in* $\mathcal{C}$*, and* B *a bialgebra (braided group) in* $\mathcal{DY}(\mathcal{C})^H_{H, \mathcal{O}(A, \overline{A})}$ *and* $H \ltimes B$ *cross product braided group with natural embedding* $i : H \hookrightarrow H \ltimes B$*. Then* $\Big((A, \overline{A}, \mathcal{R}_A), i \circ f, H \ltimes B\Big)$ *are transmutation data and* $\underline{H \ltimes B} = \underline{H} \ltimes B$ *where the left hand side is a transmutations of cross product and the right hand side is a cross product of transmutation* $\underline{H} \in \mathrm{Obj}(\mathcal{C}_{\mathcal{O}(A, \overline{A})})$ *with* B *considered as an object of* $\mathcal{DY}\left(\mathcal{C}_{\mathcal{O}(A, \overline{A})}\right)^{\underline{H}}_{\underline{H}}$*.*

THEOREM 6.2.5 *Let* $\Big((A, \overline{A}, \mathcal{R}_A), f, (H, \overline{H}, \mathcal{R}_H)\Big)$ *be transmutation data for a quasitriangular bialgebra (quantum braided group)* $(H, \underline{H}, \mathcal{R}_H)$ *in* $\mathcal{C}$*, and* $(B, \underline{B}, \mathcal{R}_B)$ *a quasitriangular bialgebra (quantum braided group) in* $\mathcal{C}_{\mathcal{O}(H, \overline{H})}$*. And let* $(H \ltimes B, \underline{H} \ltimes \underline{B}, \mathcal{R}_{H \ltimes B})$ *be generalized bosonization with natural embedding* $i : H \hookrightarrow H \ltimes B$*. Then* $\Big((A, \overline{A}, \mathcal{R}_A), i \circ f, (H \ltimes B, \underline{H} \ltimes \underline{B}, \mathcal{R}_{H \ltimes B})\Big)$ *are transmutation data and*

$$(\underline{H \ltimes B}, \underline{\overline{H} \ltimes \overline{B}}, \mathcal{R}_{\underline{H \ltimes B}})) = (\underline{H} \ltimes B, \underline{\overline{H}} \ltimes \overline{B}, \mathcal{R}_{\underline{H} \ltimes B})\,,$$

where the left hand side (resp. the right hand side) is the result of the composition of bosonization and transmutation (resp. transmutation and bosonization).

6.3. In the rest of the section we define a ribbon structure on quantum braided group in a such way that the corresponding category of modules becomes balanced and show that generalized bosonization and transmutation are compatible with ribbon structures.

A braided category $\mathcal{C}$ is called *balanced* [11] if there exists an automorphism θ of the identical functor $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$ such that

$$\theta_{X \otimes Y} = \Psi_{Y,X} \circ \Psi_{X,Y} \circ (\theta_X \otimes \theta_Y)\,. \tag{6.3.1}$$

(Existence of duals and compatibility between balancing and duality are not essential for our considerations).

DEFINITION 6.3.1 *A quantum braided group $(H, \overline{H}, \mathcal{R}_H)$ in a balanced category $\mathcal{C}$ is* ribbon *if there exists a group-like element $\gamma : \underline{1} \to H$ (i.e. $\Delta \circ \gamma = \gamma \otimes \gamma$), which satisfies the identity*

$$S_H^2 \cdot \gamma = \gamma \cdot \theta_H \tag{6.3.2}$$

This is a natural generalization of an (ordinary) ribbon Hopf algebra [35]. And, on the other hand, one can consider the identity (6.3.2) as a generalization of the axiom of spherical Hopf algebra [1]: $S^2(a) = \gamma \cdot a \cdot \gamma^{-1}$ for any $a \in H$.

PROPOSITION 6.3.1 *If $(H, \overline{H}, \mathcal{R}, \gamma)$ is a ribbon Hopf algebra in a balanced category $\mathcal{C}$ then the category $\mathcal{C}_{\mathcal{O}(H,\overline{H})}$ is also balanced with*

$$^{(\mathcal{C}_{\mathcal{O}(H,\overline{H})})}\theta = {}^{\mathcal{C}}\theta \circ (\triangleleft v)\,, \qquad \textit{where} \quad v := (u \cdot \gamma)^{-1}\,. \tag{6.3.3}$$

For quantum braided group cross product it is possible to formulate the following analog of the lemma 4.3.1 without involving of module.

LEMMA 6.3.2 *Let $(A \ltimes B, \overline{A} \ltimes \overline{B}, \mathcal{R}_{A \ltimes B})$ be a cross product of quantum braided groups $(A, \overline{A}, \mathcal{R}_A)$ in $\mathcal{C}$ and $(B, \overline{B}, \mathcal{R}_B)$ in $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. Then the 'element' $u_{A \ltimes B} : \underline{1} \to A \ltimes B$ defined in 5.1. is a product of corresponding 'elements' of A and B:*

$$u_{A \ltimes B} = u_A \cdot u_B = u_B \cdot u_A \tag{6.3.4}$$

The next proposition is a corollary of the lemmas 4.3.2 and 6.3.2.

PROPOSITION 6.3.3 *Let $\mathcal{C}$ be a balanced category, $(A, \overline{A}, \mathcal{R}_A, \gamma_A)$ be a ribbon braided group in $\mathcal{C}$ and $(B, \overline{B}, \mathcal{R}_B, \gamma_B)$ be a ribbon braided group in $\mathcal{C}_{\mathcal{O}(A,\overline{A})}$. Then the quantum group cross product $(A \ltimes B, \mathcal{R}_{A \ltimes B}, \gamma_{A \ltimes B})$ is a ribbon braided group in $\mathcal{C}$ with*

$$\gamma_{A \ltimes B} = \gamma_A \cdot \gamma_B = \gamma_B \cdot \gamma_A \tag{6.3.5}$$

LEMMA 6.3.4 *Let $(A, \overline{A}, \mathcal{R}_A)$, $(H, \overline{H}, \mathcal{R}_H)$ be quantum braided groups in $\mathcal{C}$, morphism $f : A \to H$ defines transmutation data and $(\underline{H}, \overline{\underline{H}}, \mathcal{R}_{\underline{H}})$ the transmutation. Then the 'element' $u_{\underline{H}}$ defined in 5.1. for $\underline{H}$ is a quotient of corresponding elements of H and A:*

$$u_{\underline{H}} = {u_A}^{-1} \cdot u_H = u_H \cdot {u_A}^{-1} \tag{6.3.6}$$

PROPOSITION 6.3.5 *Let $(A, \overline{A}, \mathcal{R}_A, \gamma_A)$, $(H, \overline{H}, \mathcal{R}_H, \gamma_H)$ be ribbon braided groups in $\mathcal{C}$ and morphism $f : A \to H$ defines transmutation data. Then the transmutation $(\underline{H}, \overline{\underline{H}}, \mathcal{R}_{\underline{H}})$ is a ribbon braided group with*

$$\gamma_{\underline{H}} := {\gamma_A}^{-1} \cdot \gamma_H = \gamma_H \cdot {\gamma_A}^{-1} \tag{6.3.7}$$

6.4. More concrete applications of the theory that is developed here can be find in [4] where we develop a fully braided analog of Faddeev-Reshetikhin-Takhtajan construction of quasitriangular bialgebra $A(X, R)$. For given pairing C factor-algebra $A(X, R; C)$ which is a dual quantum braided group is built. Corresponding inhomogeneous quantum group is obtained as a result of generalized bosonization. We define a fully braided analog of Jurčo construction of first order bicovariant differential calculus.

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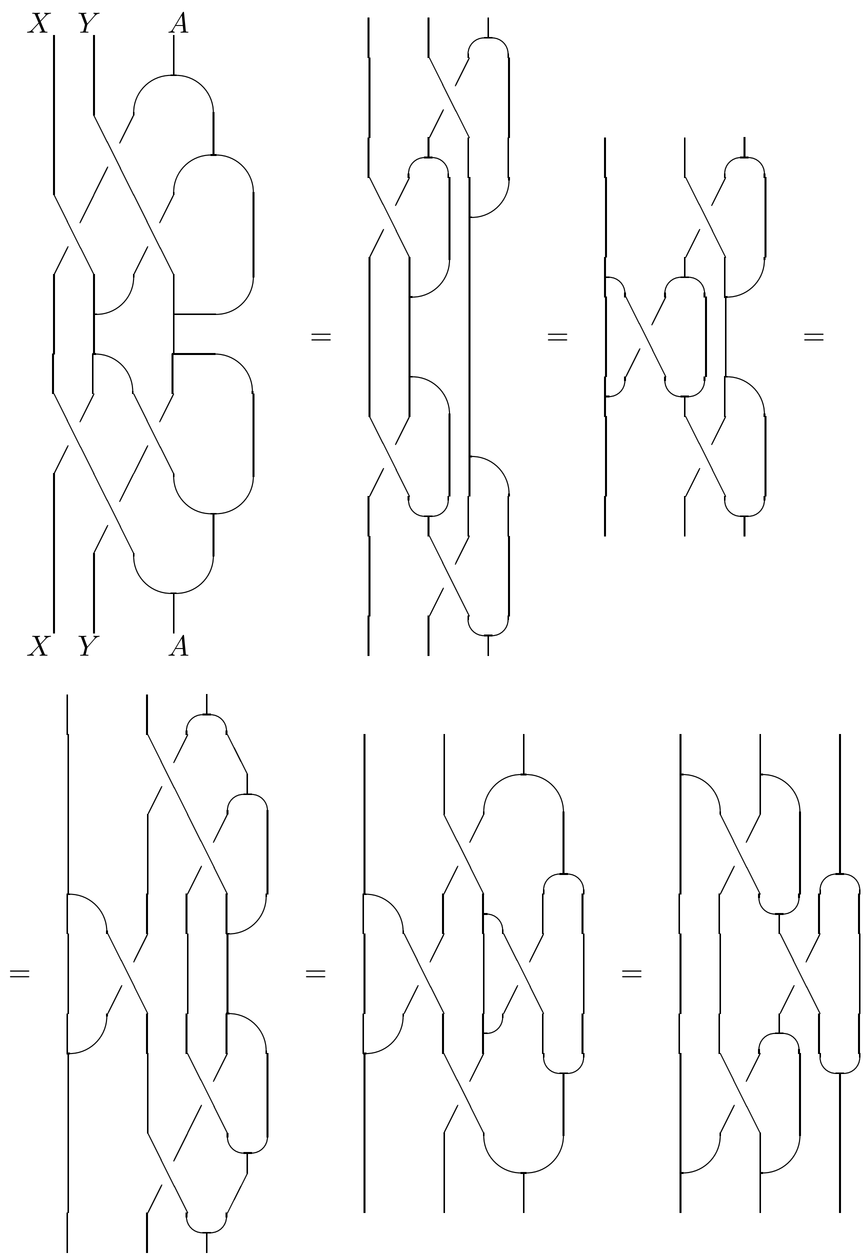


Figure 18: Proof of the crossed module axiom for tensor product of crossed modules:
We use coherence and associativty (the first, the third, the fifth equalities), the crossed module axiom for X and Y (the second and the fourth equalities respectively).

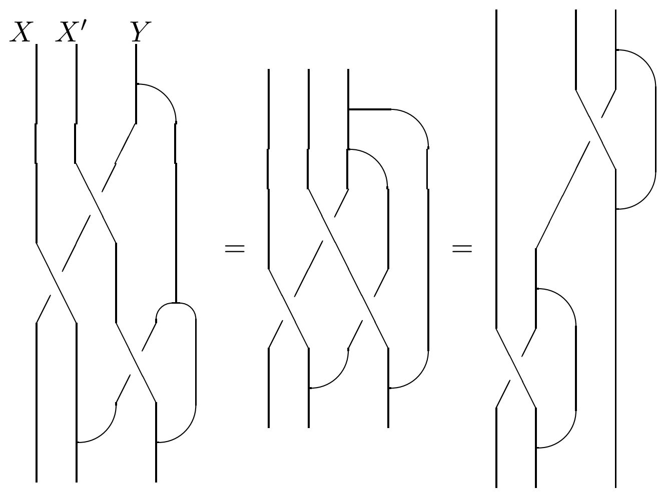


Figure 19: Proof of the hexagon identities for Ψ^A

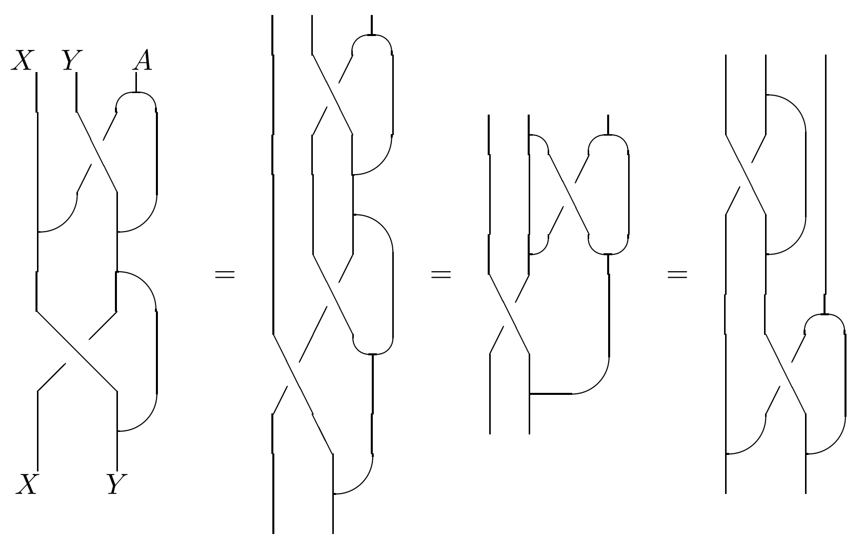
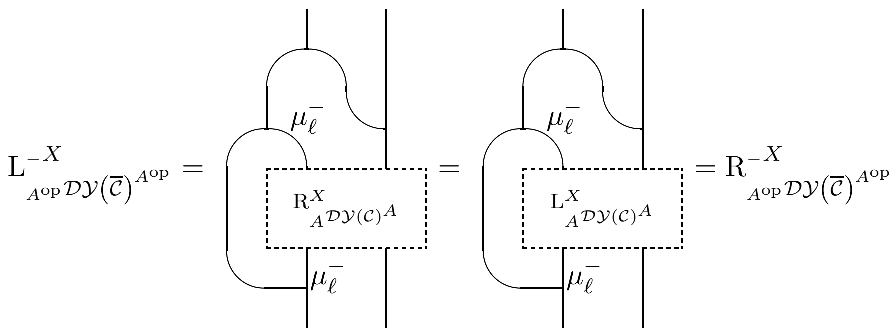
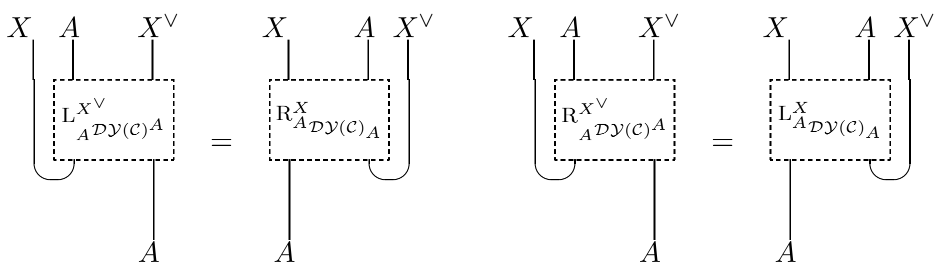


Figure 20: Braiding in $\mathcal{DY}(\mathcal{C})_A^A$ is a module map:
The first, the second and the third equalities are the module, crossed module and comodule axioms respectively.



a) If $X \in \mathrm{Obj}({}_A\mathcal{DY}(\mathcal{C})^A)$ then ${}_-X \in \mathrm{Obj}({}_{A^{\mathrm{op}}}\mathcal{DY}\left(\overline{\mathcal{C}}\right)^{A^{\mathrm{op}}})$.



b) If $X \in \mathrm{Obj}({}^A\mathcal{DY}(\mathcal{C})_A)$ then $X^\vee \in \mathrm{Obj}({}_A\mathcal{DY}(\mathcal{C})^A)$.

Figure 21: Constructions of new crossed modules. Proof of compatibility conditions.

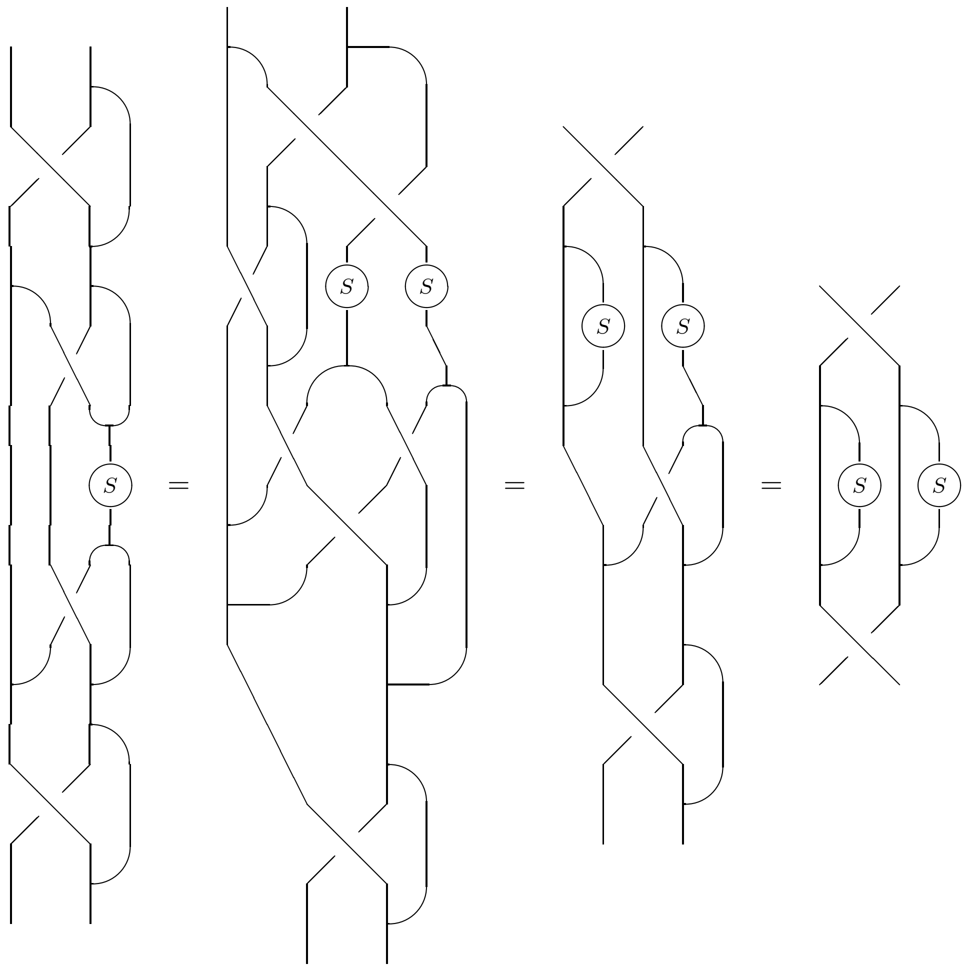


Figure 22: Proof of the identity ${}^{(\mathcal{DY}(\mathcal{C})^H_H)}\Psi \circ S^2 \circ {}^{(\mathcal{DY}(\mathcal{C})^H_H)}\Psi = {}^{\mathcal{C}}\Psi \circ (S^2 \otimes S^2) \circ {}^{\mathcal{C}}\Psi$: The first equality uses the facts that ${}^{(\mathcal{DY}(\mathcal{C})^H_H)}\Psi$ is a comodule morphism, antipode is anti-algebra morphism, the bialgebra and module axioms. The second uses the fact that antipode is anti-coalgebra morphism and the antipode axiom. The third uses the facts that ${}^{(\mathcal{DY}(\mathcal{C})^H_H)}\Psi$ is a module morphism, antipode is anti-coalgebra morphism and the antipode axiom.

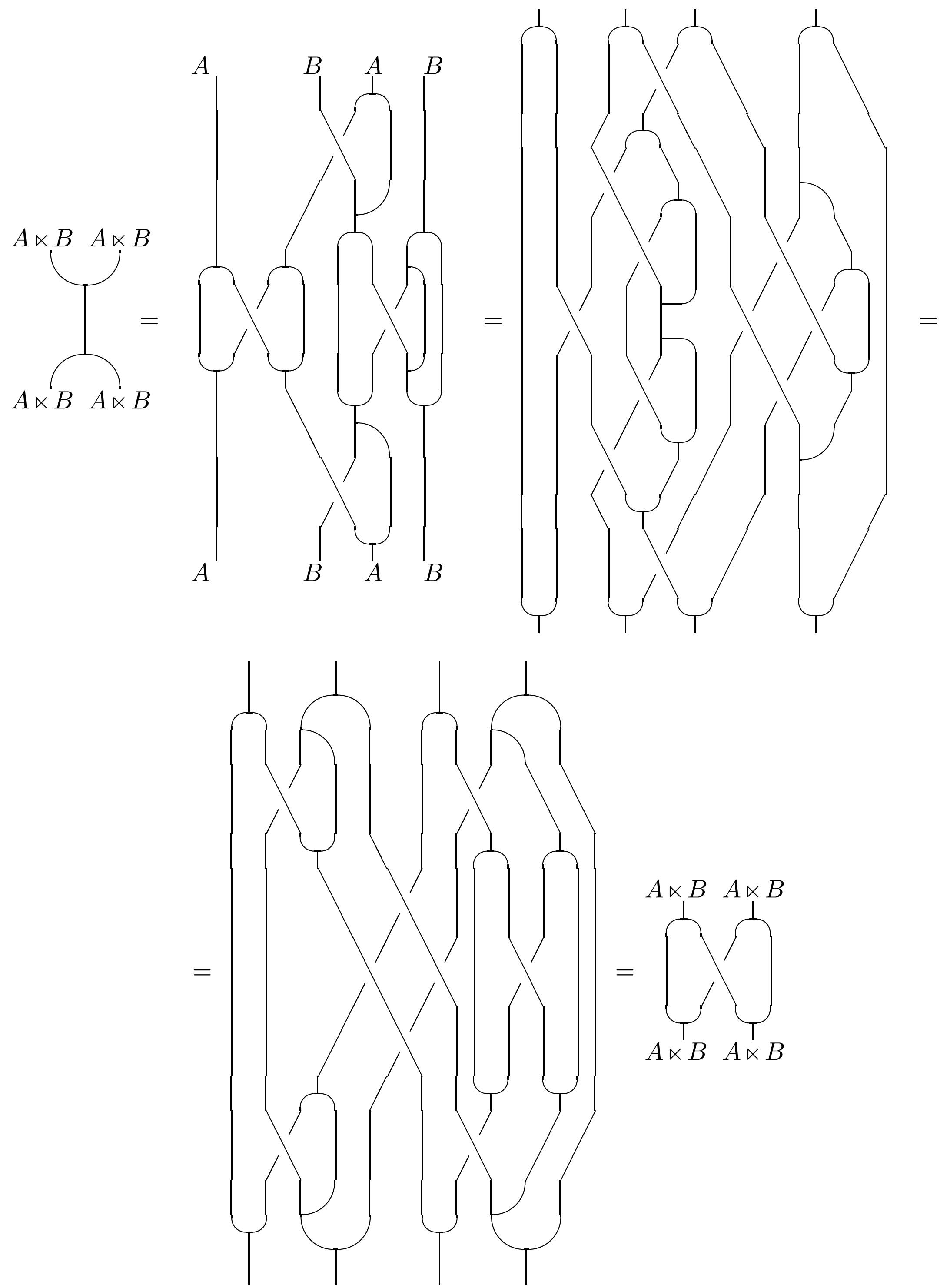


Figure 23: Proof of the bialgebra axiom for $A \ltimes B$:
The first equality follows from the bialgebra axiom for A and B. The second uses the fact that B is a coalgebra in $\mathcal{C}_A$ and an algebra in $\mathcal{C}^A$. The third uses the crossed module axiom. The last is the bialgebra axiom for A.

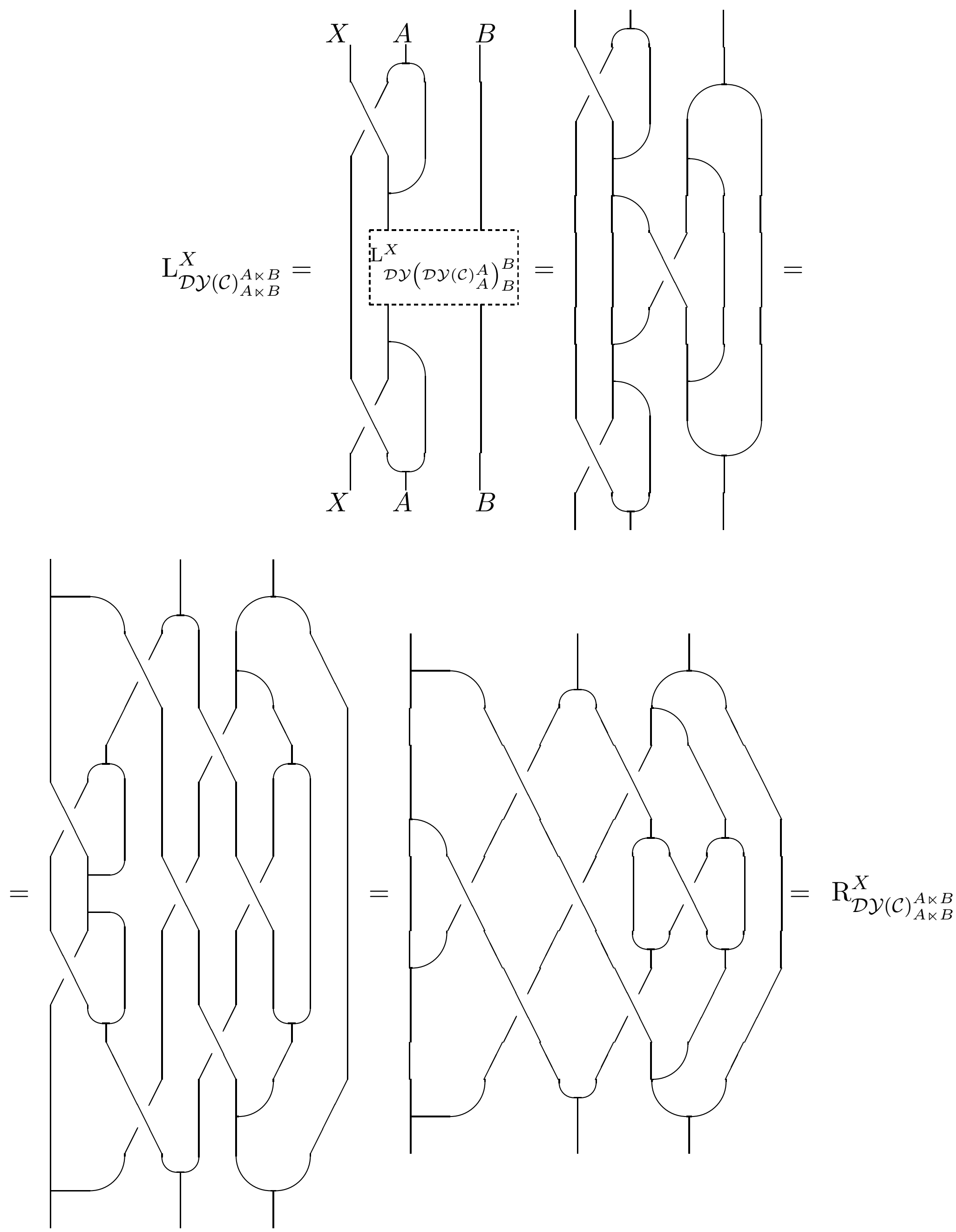


Figure 24: Proof of the crossed module axiom over cross product $A \ltimes B$:
The second equality is the crossed module axiom for X over B. The third uses the fact that B-action is A-module and B-coaction is A-comodule morphisms respectively. The 4th is the crossed module axiom for X over A. The 5th uses the bialgebra axiom for A.

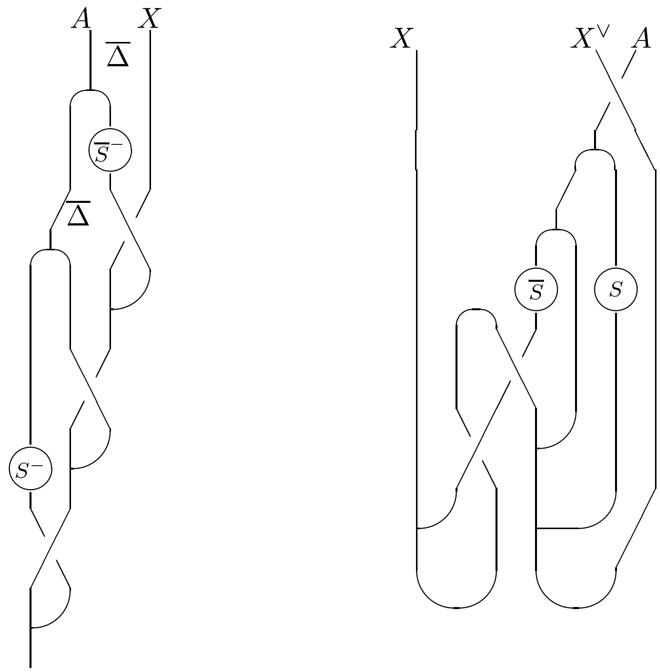


Figure 25: Proof of the identities in Fig.14.

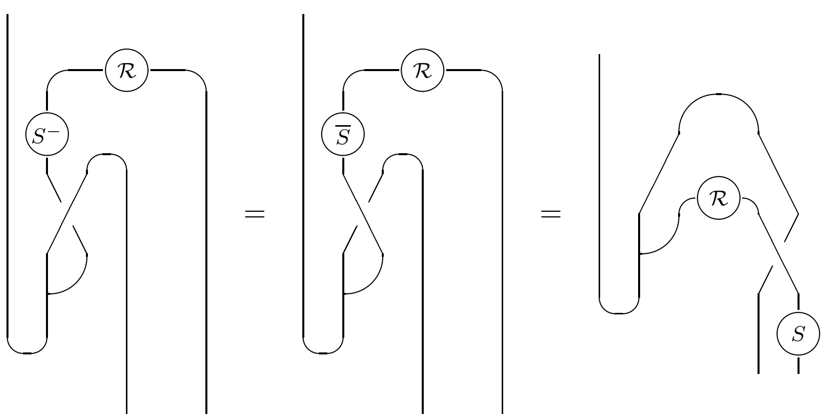


Figure 26: Comodule structures on $({}^\vee X)^{\mathcal{R}}$ and ${}^\vee(X^{\mathcal{R}})$ are the same: The first identity follows from the first identity in Fig.14.

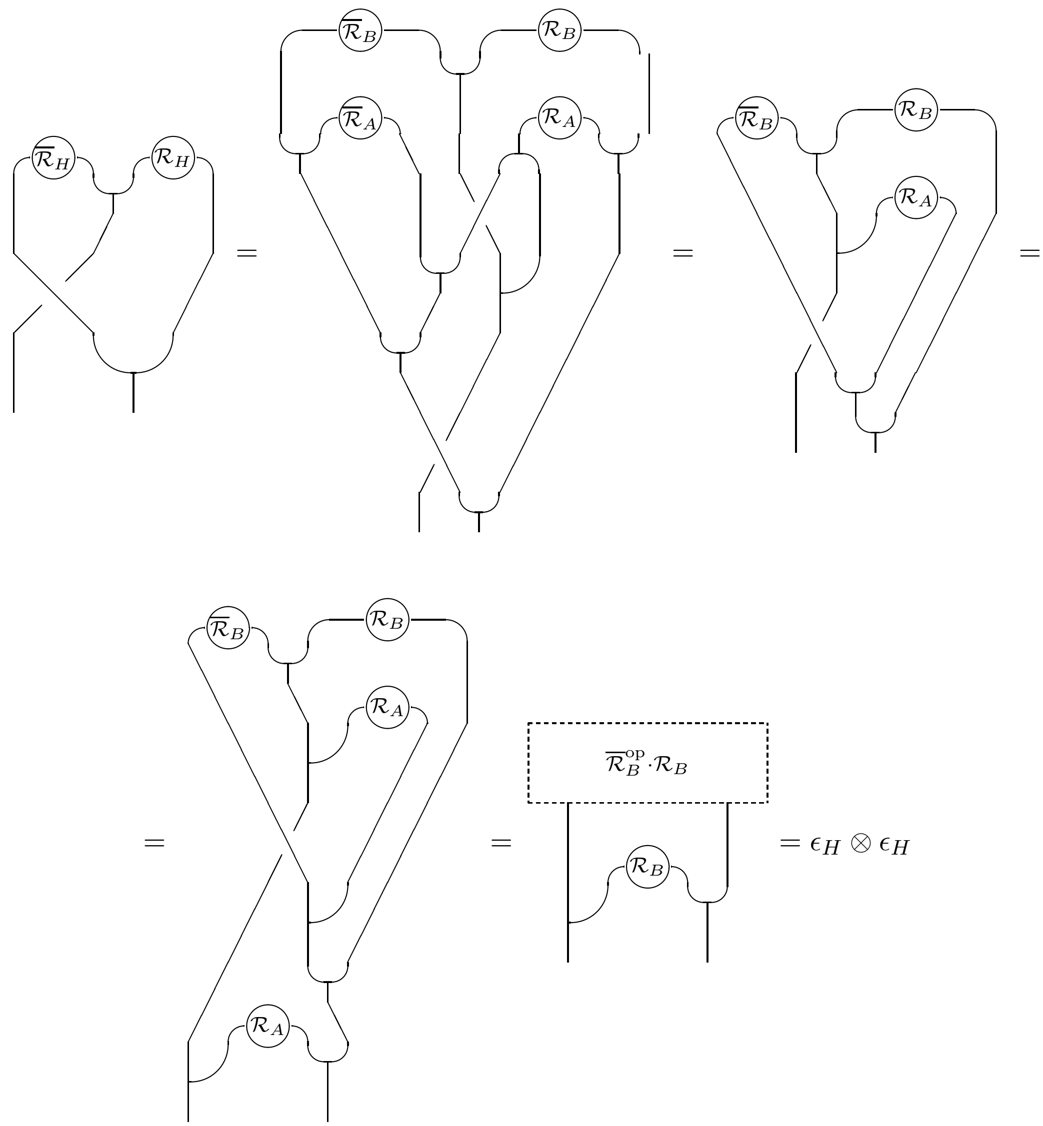
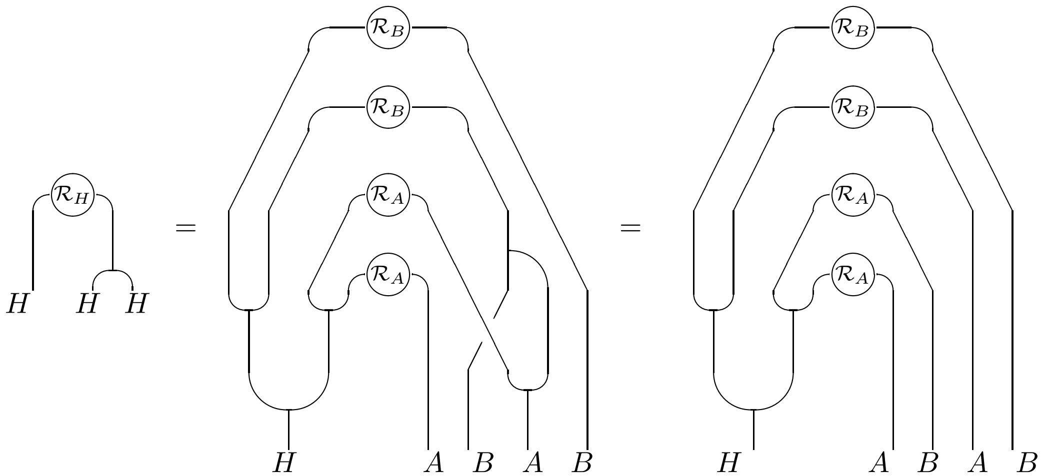
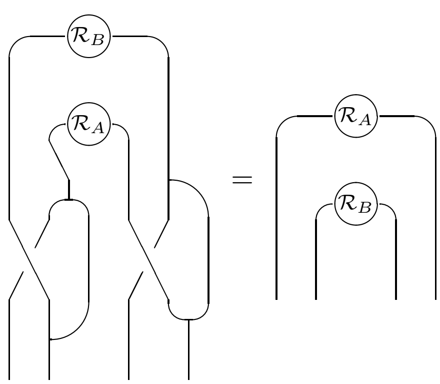


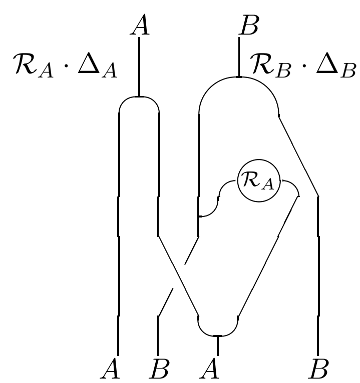
Figure 27: Proof of the identity $\overline{\mathcal{R}}^{\mathrm{op}}_{A \ltimes B} \cdot \mathcal{R}_{A \ltimes B} = \epsilon_H \otimes \epsilon_H$:
The first and the third identities use the formula for multiplication in $\overline{A} \ltimes \overline{B}$ and $A \ltimes B$ respectively. The second and the last use the fact that $\overline{\mathcal{R}} \cdot \mathcal{R} = \epsilon \otimes \epsilon$ for A and for B respectively.



a) $\mathcal{R}_H$ is an algebra-coalgebra copairing



b)The auxiliary identity



c) $\overline{\Delta}^{\mathrm{op}} \cdot \mathcal{R} = \mathcal{R} \cdot \Delta$

Figure 28: Proof of the quantum braided group axioms:
In the part a): the first equality use the fact that $\mathcal{R}_A$ and $\mathcal{R}_B$ are algebra-coalgebra pairings. The second follows from the auxiliary identity from the part b) of this figure.